A TREATISE

ON

GEOMETRICAL OPTICS.

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BY

R. A. HERMAN, M.A.

Robert Alfred

FELLOW AND LECTURER OF TRINITY COLLEGE, CAMBRIDGE.

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INTRODUCTION.

IN this treatise one of my aims has been to introduce a new method of determining the properties of a symmetrical optical instrument, in which the angle of divergence of a small pencil, rather than any coordinate of its origin, has been adopted as the leading feature. By this method, which is implicitly contained in Cotes's well-known theorem of the apparent distance, all the cardinal properties, notably the power of the system, follow immediately from the application of a simple geometrical formula. Although that theorem dates from the middle of the eighteenth century, yet the present method needs the help of Helmholtz's theorem to render it complete; hence the priority of the proof of the existence of the cardinal points rests with Gauss. But the analytical method adopted by the latter tends, in my opinion, to disguise the essential fact that geometrical foci exist only as a first approximation; and in any particular case a lengthy calculation is necessary to obtain the final convergent of the continued fraction on which his results depend. The simplification that results from the use of the angle of divergence will, I think, be clear on reference to Chapter V. and to the chapter on Aberration.

I have attempted to obviate as far as possible the difficulties of sign that arise in connection with the directions in which distances are to be measured by the convention that all lengths on the same straight line are to be treated as algebraical quantities, and, where an origin of light is the origin of measurement, by taking the direction in which the light travels as positive.

The oblique incidence of small pencils is really dealt with by geometry rather than by the use of the characteristic function;

and similarly in the many theorems due to Clerk Maxwell, it is the reduced path between two points, and not that function which forms the basis of the work. In fact the characteristic function, regarded as defined by its differential equation, is only applied to one optical problem, namely that in Article 185.

I must express my thanks to my friends Mr E. G. Gallop, Mr H. C. Robson, and Mr A. N. Whitehead for their criticism and assistance in various parts of the book. I am under deep obligation to Mr H. L. Aldis for the formulae contained in the earlier part of Chapter XIV., which are the first to give completely the distortion or extra-axial aberration in any symmetrical instrument; but in place of his method, which is trigonometrical, I have availed myself of the angle of divergence, as being more consonant with the plan of this work.

References have been given when possible to the authorities for the various theorems, and the collected papers of Clerk Maxwell and Cayley have been largely utilised. The examples are taken for the most part from College and University Examination papers.

R. A. HERMAN.

TRINITY COLLEGE,
CAMBRIDGE.
September, 1900.

CONTENTS.

CHAPTER I.

RAYS OF LIGHT.

CHAPTER II.

REFRACTION.

CHAPTER III.

GEOMETRICAL FOCI.

CHAPTER IV.

LENSES.

CHAPTER V.

COAXIAL REFRACTING SURFACES.

CHAPTER VI.

OPTICAL INSTRUMENTS.

CHAPTER VII.

DISPERSION AND ACHROMATISM.

CHAPTER VIII.

ABERRATION.

CHAPTER IX.

ILLUMINATION.

CHAPTER X.

GENERAL THEOREMS.

CHAPTER XI.

CHARACTERISTIC FUNCTION.

CHAPTER XII.

CAUSTICS.

CHAPTER XIII.

HETEROGENEOUS MEDIA.

CHAPTER XIV.

DISTORTION.

CHAPTER I.

RAYS OF LIGHT.

1. THE science of Geometrical Optics, like the deductive part of other physical sciences, is based on an abstraction, namely a ray of light.

A luminous body is believed to be visible to the eye by means of the vibrations it excites in a hypothetical medium pervading all space, and known as the ether. These vibrations travel through the ether, and finally affect the retina of the eye in some manner, giving rise to the sensation of brightness. The vibrations are affected by the matter which may also pervade the space through which they travel. If it check their transmission, it is opaque; if it allow of their transmission, it is transparent. Also the velocity of propagation of the vibrations is affected by the nature of the material.

Now although each such disturbance of the ether is itself the centre of new disturbances spreading out in all directions, yet it is a matter of constant experience that a small opaque obstacle placed in a homogeneous material medium in the straight line between the eye and the source of light completely destroys the sensation of light. This fact may be explained by the principles of Physical Optics; and both the explanation and the experiment require us to assume that the dimensions of the obstacle are many times those of the wave-length of light, a quantity varying from 4×10^{-5} to $7{\cdot}6 \times 10^{-5}$ centimetres *.

The shadow of an obstacle of finite dimensions, which is formed by a very small origin of light, is practically its geometrical projection; and we are inevitably led to the idea that the light travels from its origin in straight lines. These straight lines we

* Glazebrook, *Physical Optics*, Chap. II.

call rays; the collection of rays by which a source of light is visible to the eye is called a pencil, and may be thought of as a small cone having its vertex at the source.

When a pencil of light, travelling in one homogeneous medium, is incident on the surface of separation from another medium, part of the light is scattered, part undergoes a change of direction in the first medium and is reflected, while part may enter the second medium in a different direction and is said to be refracted.

The alterations in the form of a pencil of light when it is reflected, or when it is refracted in passing from one medium to another, are given by well-established laws, which are expressed in terms of rays; although no ray can possibly exist except in conjunction with other rays.

The deductions from these laws, with the assumption of rectilinear rays in a homogeneous medium, form the subject of geometrical optics.

REFLECTION OF LIGHT.

2. The fundamental law of reflection is that *the incident and reflected rays lie in the same plane with the normal to the reflecting surface at the point of incidence, and make equal angles with the normal on opposite sides.*

The plane containing the incident ray and the normal is called the plane of incidence; it will therefore also be the plane of the normal and the reflected ray, or the plane of reflection.

The acute angle between the incident ray and the normal is called the angle of incidence; the acute angle between the reflected ray and the normal the angle of reflection; and these angles are equal.

The deviation is the angle through which the direction of the reflected ray has been turned from that of the incident ray; it is therefore in this case the supplement of twice the angle of incidence.

3. *When a ray of light is reflected at any surface, the incident and reflected rays make the same angle with any plane through the normal; and the projections of the rays on such a plane obey the law of reflection.*

Let O be the point of incidence, take equal distances OP, OQ

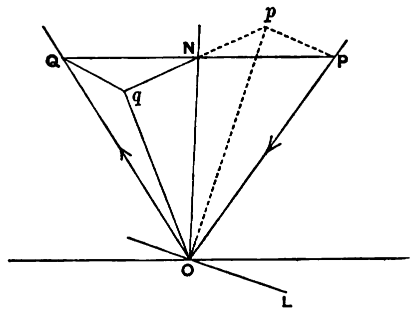


Fig. 1.

on the incident and reflected rays respectively; then since the angle of incidence is equal to that of reflection, OP and OQ have the same projection ON on the normal to the reflecting surface; also N bisects PQ. Let Op, Oq be their projections on any plane through the normal.

It is obvious that the right-angled triangles PpN, QqN are equal in all respects.

Hence the right-angled triangles OpP, OqQ are equal in all respects. The angles POp, QOq made by the rays with the arbitrary normal plane pOq are therefore equal.

Since the normal to this plane is an arbitrary line OL in the reflecting plane, *i.e.* the tangent plane to the reflecting surface, this theorem can also be expressed in the form:—

The direction OQ of the reflected ray makes with any direction in the reflecting plane the same angle as the direction PO of the incident ray.

Again, since $pN = qN$, and ON is perpendicular to pq, the projections Op, Oq on the plane through the normal obey the law of reflection, and make equal angles with the normal.

4. Direction-Cosines of the reflected ray.

Let (l, m, n) be the direction-cosines referred to any rectangular axes of the incident ray PO, (l', m', n') those of the reflected ray OQ, taken in each case in the direction in which the light is moving; let (λ, μ, ν) be the direction-cosines of the normal ON, and ϕ the angle of incidence and reflection.

Since N is the middle point of PQ, the algebraic sum of the projections of OP and OQ on any straight line is twice the projection of ON on that line. The projections of OP and PO are of opposite signs. If the line be taken in the reflecting plane, ON has no projection on it, and therefore OQ and PO have equal projections, or, in other words, make equal angles with any line in the reflecting plane.

Projecting however on the axes of reference, we obtain

$$\left.\begin{aligned} l' - l &= 2\lambda \cos\phi \\ m' - m &= 2\mu \cos\phi \\ n' - n &= 2\nu \cos\phi \end{aligned}\right\},$$

since $$ON/PO = ON/OQ = \cos\phi.$$

Moreover $\cos\phi = l'\lambda + m'\mu + n'\nu = -(l\lambda + m\mu + n\nu)$.

5. Reflection at a Plane.

If a pencil of rays diverging from a point be reflected at a plane, the reflected rays will form a pencil diverging from another point, which is called the image of the origin of light. Let Q be the origin of light, and let QM be drawn perpendicular to the plane, and produced to a point Q' such that QQ' is bisected in M, then the lines QR, $Q'R$ will obviously lie in the same plane with the normal to the plane at R, and make equal angles with it.

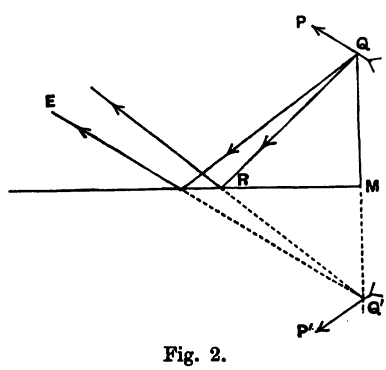


Fig. 2.

Hence, if QR be any incident ray, the reflected ray is in the direction of $Q'R$ produced.

The reflected rays therefore form a pencil, which appears to diverge from Q'.

Moreover, since the normals at the points of incidence are all parallel, the angle between any two reflected rays is equal to the angle between the incident rays, and therefore the angle of divergence of the incident pencil is unaltered.

The image of any object PQ will be an equal object $P'Q'$, formed by the images of the different points of PQ. The image is "perverted," *i.e.* the image of printed characters is upright, but appears to run from right to left.

The actual size of the image is the same as that of the object, but as its distance from the eye may differ from that of the object, they will subtend different angles at the eye, and will therefore appear to be unequal.

6. Field of View in a plane mirror.

By exchanging the positions of Q and E in § 5, it is obvious that any ray,

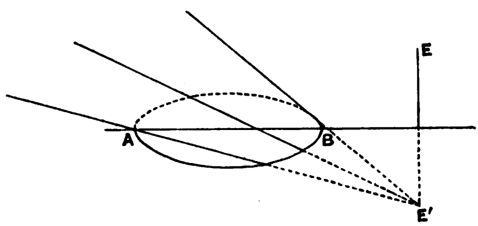


Fig. 3.

which after reflection at a limited plane mirror AB passes through the eye E, must have been incident in a direction passing through the image E' of the eye in the mirror.

All objects then that are visible by reflection must lie within the solid angle bounded by lines from E' to the circumference of the mirror.

7. Two parallel plane mirrors.

If a luminous object be placed between two parallel plane mirrors, there will be formed two infinite series of images by successive reflections. These images will all be situated on the common normal to the two mirrors through the luminous object, and will be equal in size to it.

Let Q be the object, and a and b its distances from the planes A and B respectively, let c be the distance AB between the planes.

Then if Q_1 be the image formed by one reflection at the mirror A, $QQ_1 = 2a$; and if Q_2 be the image of Q_1 formed by a

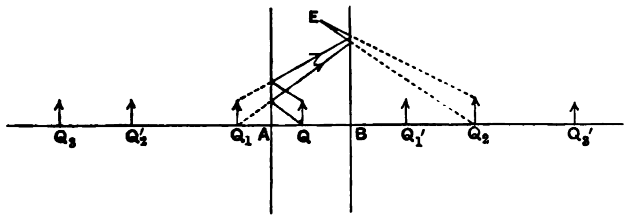


Fig. 4.

reflection at B, $BQ_2 = BQ_1$, and hence $QQ_2 = b + b + 2a = 2c$. So also $Q_1Q_3 = 2c$, treating Q_1 as an origin of light, and Q_3 as its image after two reflections of the light from it.

Hence the distance between successive images formed by an even number of reflections is $2c$, and similarly the distance between successive images formed by an odd number of reflections is $2c$.

We can find in the same way the positions of the other series of images, Q_1', Q_2'..., beginning with the image formed by reflection at B; and since $Q_1Q_1' = 2c$, the bright point and the two series of even images form one infinite series of points at distance $2c$ apart, while the odd images also form another such series.

The course of a pencil of light from any image to the eye is found by joining that image to the eye, by joining the points, where these lines cut that mirror in which this image is formed, to the corresponding points of the previous image, then by joining the points where these lines cut the other mirror to the next preceding image, and so on till the object is reached. In this way it is clear that the distance actually traversed by the light is equal to the distance from the eye of the image viewed; and that the angle of convergence of the pencil is the same as if that image were viewed directly. Hence the successive images subtend diminishing angles at the eye; and as at every reflection some light is lost, an explanation is afforded of the apparent decrease in size and brightness of the row of images of any luminous object, which is seen in two looking-glasses opposite to each other.

The images formed by an odd number of reflections are *perverted*; those formed by an even number are similar to the object.

8. The Kaleidoscope.

A luminous object placed between two intersecting plane mirrors will give rise to two series of images, whose numbers will depend on the position of the object and the angle between the mirrors.

Let a plane through a bright point Q perpendicular to the line of intersection of the mirrors meet that line in O and the mirrors in the lines OA, OB.

The distances of all the images from O are equal to that of Q, and they therefore lie on a circle.

Let α, β be the arcual distances QA, QB, and γ the arc AB, all reckoned in circular measure.

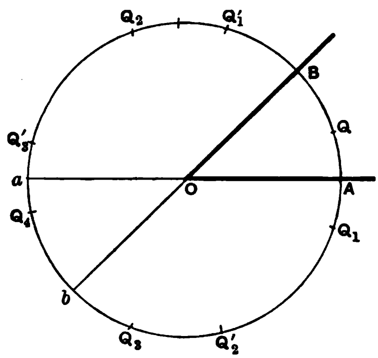


Fig. 5.

Then if Q_1 be the image of Q in the mirror OA, Q_2 of Q_1 in OB, Q_3 of Q_2 in OA and so on, the arc $QQ_1 = 2\alpha$, and the arc $QQ_2 = QB + BQ_2 = QB + BQ_1 = 2\gamma$.

Hence the arc Q_1Q_3 is also 2γ, and this is the angle between successive pairs of images formed by an even or odd number of reflections.

Hence $QQ_{2n} = 2n\gamma$ taken positively, and $QQ_{2n+1} = 2\alpha + 2n\gamma$ taken negatively round the circle.

The number of images in the series is finite, and the last image falls inside the arc ab behind both mirrors, for it is plain that rays which after reflection at one mirror apparently diverge from such a point cannot possibly meet the other mirror.

It is obvious from the equations $AQ = \alpha$, $BQ_1 = \gamma + \alpha$, $AQ_2 = 2\gamma + \alpha$, and so on, that the distance of Q_n from the mirror in which the next image is to be formed is $n\gamma + \alpha$. As long as this angle is less than π, the next image will be formed, but reflections will cease when this angle exceeds π.

Hence the number of images in this series is the integer next above $(\pi - \alpha)/\gamma$.

We may also find the positions of the other series of images $Q_1'Q_2'\ldots$ formed by rays reflected first at the mirror OB, and the number in this series is the integer next above $(\pi-\beta)/\gamma$.

Since $QQ_1 = 2\alpha$ and $QQ_1' = 2\beta$, it follows that $Q_1Q_1' = 2\gamma$, and therefore the bright point and the even images form a series of points at arcual distances 2γ from each other, running to either side of Q; and the odd images form another such series.

If the angle between the mirrors be π/m, where m is integral, then the polygon in which each side cuts off an arc $2\pi/m$ will be closed; and therefore either the last two images with an even suffix, or the last two images with odd suffix, coincide inside ab, according as m is even or odd. In either case the number of images formed is $2m-1$.

9. Visibility of the last image.

The last images formed, which fall inside ab, will not be seen unless the eye lie within certain parts of the angle AOB.

For if Q_{2n} be the last image of the first series, then since this is an image in the mirror B, all the rays that appear to proceed from it after being reflected in B, will fall within an angular distance from B less than the angle bOQ_{2n}. That is, the distance of the eye from B must not exceed $\pi+\beta-2n\gamma$. Similarly the necessary position of the eye to perceive Q_{2n+1}, if that be the last image, can be found.

In the case of $\gamma=\pi/m$, the last image is always seen, since it belongs to both series of images, and is therefore visible in one or other of the mirrors, according to the position of the eye.

10. Example.

If the angle between two plane mirrors be $2\pi/n$, where n is an odd integer, the number of images formed will be n for all positions of the object; but if the arcual distance of the eye from both mirrors exceed that of the object from the bisector of the angle, only $n-1$ are visible.

With the notation of § 8, let a be the distance of the bright point Q from the nearer mirror; then $a/\gamma<\frac{1}{2}$, and therefore $(\pi-a)/\gamma>(n-1)/2$; that is, the integer next above $(\pi-a)/\gamma$ is $(n+1)/2$. This then is the number of images in the first series.

Similarly since $\beta/\gamma>\frac{1}{2}$, $(\pi-\beta)/\gamma<(n-1)/2$, and the number of images in the second series is $(n-1)/2$.

(i) Let $(n+1)/2$ be an even integer, $(n-1)/2$ odd; then each of these last images is an image in B.

Also $QQ_{(n+1)/2}=(n+1)\pi/n=\pi+\frac{1}{2}\gamma$; and therefore the distance of this image from b is $\beta-\frac{1}{2}\gamma$. In order to see it, the arcual distance of the eye from B must be less than $\beta-\frac{1}{2}\gamma$, *i.e.* less than the distance of the bright point from the bisector of the angle γ.

Again, $QQ'_{(n-1)/2} = 2\beta + \frac{n-3}{2}\frac{2\pi}{n} = \pi - \frac{3}{2}\gamma + 2\beta$, and the distance of this image from b is $\frac{3}{2}\gamma - \beta$ or $\frac{1}{2}\gamma + a$. Hence to see this the distance of the eye from B must be less than $\frac{1}{2}\gamma + a$, or from A must be greater than $\frac{1}{2}\gamma - a$, *i.e.* than the distance of the bright point from the bisector.

When the distance of the eye from B is less than the distance of Q from the bisector, all n images are visible; when the distance of the eye from A is less than this quantity, the last image in each series is invisible; if the eye be in an intermediate position, the last image of the first series is invisible.

(ii) Let $(n+1)/2$ be an odd integer, $(n-1)/2$ even; then the last images are both images in A.

Also $QQ_{(n+1)/2} = 2a + \frac{n-1}{2}\frac{2\pi}{n} = \pi + 2a - \frac{1}{2}\gamma$; and the distance of this image from A is $\frac{1}{2}\gamma - a$. To see this the distance of the eye from A must be less than $\frac{1}{2}\gamma - a$, *i.e.* than the distance of Q from the bisector.

So $QQ'_{(n-1)/2} = \frac{n-1}{2}\frac{2\pi}{n} = \pi - \frac{1}{2}\gamma$, and the distance of this image from A is $a + \frac{1}{2}\gamma$. The eye therefore must be at a distance from A less than $a + \frac{1}{2}\gamma$, or at a distance from B greater than $\frac{1}{2}\gamma - a$. The total number of images seen therefore is, as before, n, $n-1$, $n-2$, as the distance of the eye from A is less than $\frac{1}{2}\gamma - a$, greater than $\frac{1}{2}\gamma - a$ and less than $\frac{1}{2}\gamma + a$, or greater than $\frac{1}{2}\gamma + a$.

11. Path of a ray reflected between two intersecting plane mirrors.

Let a ray of light be reflected in one plane between two mirrors inclined to each other at an angle γ.

Denote the angle which the directions of the incident and reflected ray at the first point of incidence A make with AO by θ_1; and the angles between the directions of the incident and reflected rays at B and BO by θ_2, and so on. These angles may be acute (as at A), or obtuse (as at B in the figure).

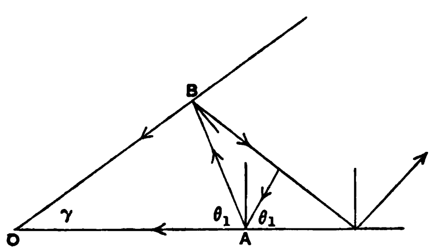


Fig. 6.

Then we have the equations $\theta_2 - \theta_1 = \gamma$, $\theta_3 - \theta_2 = \gamma$, &c. Hence it follows that $\theta_3 - \theta_1 = 2\gamma$, or since these angles are measured from the same line AO, that the deviation of a ray after successive reflection at two mirrors is twice the angle between them.

Again, by addition of the equations above, $\theta_r - \theta_1 = (r-1)\gamma$; if any value of r make θ_r equal to $\pi/2$, then the rth incidence will be normal, and the ray will retrace its path.

The reflections will cease as soon as θ_r exceeds the value $\pi - \gamma$, since at that reflection the reflected ray, travelling outwards, will make an angle less than γ with one mirror, and will therefore not meet the other again.

The number of reflections is therefore r, where r is the integer next above $(\pi - \theta_1)/\gamma$.

If a circle be described, with O as centre, to touch the incident ray, then all parts of its path will continue to be tangents to the circle, and the path may be thus constructed geometrically.

When the path of the ray is not in the principal plane normal to both mirrors, then it may be constructed from the fact that all parts of the path make the same angle with the line of intersection of the mirrors; and that its projections on the principal plane behave as a ray of light, and are therefore all tangents to the same circle. It follows from these considerations that the successive parts of the path are generators alternately of opposite systems of a hyperboloid of revolution of one sheet.

12. Application of Spherical Trigonometry.

Through the centre of any sphere draw radii in the same directions as the incident and reflected rays and the normal at the point of reflection, meeting the sphere in P, Q, N respectively.

These points lie on a great circle, and the arcs NP, NQ are supplementary. The arc PQ is bisected at right angles by the great circle, having N as pole; and therefore P, Q are equidistant from any point L on that circle. In other words, the two parts of the path make equal angles with any line in the reflecting plane. Moreover, if we take LM a quadrant, NM will represent any plane through the normal, and if LP, LQ cut NM in p, q respectively, these points represent the projections of the parts of the path on that plane, and obviously Np, Nq are supplementary.

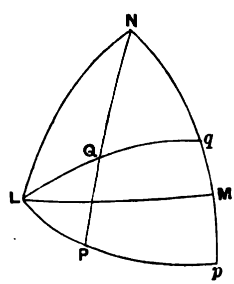


Fig. 7.

13. *To find the deviation when a ray is reflected between two plane mirrors, the path not lying in a principal plane.*

If a ray be reflected successively between two plane mirrors

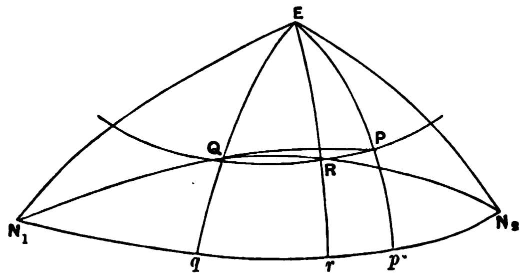


Fig. 8.

inclined at an angle γ, let N_1 and N_2 represent on the sphere the normals to these planes drawn towards the sides at which reflection takes place, so that $N_1N_2 = \pi - \gamma$; and let E represent the line of intersection, so that $N_1E = N_2E = \pi/2$.

If P, Q, R represent respectively an incident ray and its direction after one and two reflections, then

$$N_1P + N_1Q = \pi; \quad N_2Q + N_2R = \pi.$$

Let $EP = EQ = ER = \theta$, where θ is the angle which the ray throughout makes with the line of intersection of the planes, and let the arc PR, which is the deviation, be denoted by D. Then bisecting the isosceles triangle PER, we have

$$\sin \tfrac{1}{2} D = \sin\theta \sin \tfrac{1}{2} PER.$$

But if p, q, r be the projections of P, Q, R on the principal plane, $N_1p + N_1q = \pi$, $N_2q + N_2r = \pi$;

and hence $\qquad PER = pr = N_1p + N_2r - N_1N_2 = 2\gamma,$

and therefore $\qquad \sin \tfrac{1}{2} D = \sin\theta \sin\gamma.$

Similarly the deviation after $2n$ reflections is

$$2\sin^{-1}(\sin\theta \sin n\gamma),$$

since the deviation of the projection of the ray on the principal plane is $2n\gamma$.

The number of reflections is finite, and they will cease as soon as the point representing the ray after reflection at one plane approaches within $\tfrac{1}{2}\pi$ of the normal to the other plane.

EXAMPLES.

1. A mirror of length l hangs forwards from a wall making an angle a with it. Shew that a person whose height is h can just see the whole of himself in the mirror if his distance from the wall be

$$2l \sin a\,(h \cos a - l)/(h \cos a - 2l),$$

and the height of the point of contact of the mirror with the wall be

$$(h - 2l \cos a)\,(h \cos a - l)/(h \cos a - 2l).$$

2. A cylinder polished internally has a bright point at the centre of its base. A screen is placed perpendicularly to the cylinder at a distance above it equal to its height; shew that the appearance on the screen is a bright circle of radius equal to twice that of the cylinder, surrounded by rings of equal breadths and of successively decreasing brightness.

Shew further that if the height of the screen above the cylinder be $\frac{3}{2}$ times the height of the cylinder, the appearance on it is a series of alternately brighter and darker rings, of breadths equal to the diameter and the radius of the cylinder respectively.

3. Two parallel plane mirrors of height h are at a distance c apart, and a bright point is placed at the foot of one of them. A screen is placed perpendicularly to them at height R above their highest points; shew that if $R>h$ there will be on the screen alternately bright and dark bars of breadths $c(1+R/h)$ and $c(R/h-1)$ respectively. If $R<h$ but $>\frac{1}{3}h$, there will be bars of breadths $b(1-R/h)$ and $b(3R/h-1)$ respectively, the brightness of which will alternate.

4. If the sides of an equilateral triangle ABC be reflectors, and a luminous point be placed at an internal point whose distances from the sides are a, β, γ; then each of the images formed by continued reflections at the sides in order, beginning with BC, is at a distance from one or other of the sides equal either to β or to $a+\gamma$.

5. A bright point P whose trilinear coordinates are a, β, γ is placed within a reflecting triangle ABC; and $PQRSP$ is the path of a ray which returns to P after reflection at the sides BC, CA, AB of the triangle in order. Prove that the length of the path is $2l$, where

$$l^2 = a^2 + \beta^2 + \gamma^2 + 2\beta\gamma \cos A + 2\gamma a \cos(A - C) + 2a\beta \cos C.$$

Hence shew that points for which l has a constant value lie on two parallel straight lines; and find when l is a minimum.

6. If the angle of a hollow cone polished internally be any sub-multiple of two right angles, a cylindrical pencil of rays incident parallel to the axis will after a certain number of reflections be a cylindrical pencil parallel to the axis and of the same diameter as the incident pencil.

7. If the angle between two plane mirrors be 50°, and a luminous object lie within 5° of the bisector, 8 images will be formed, of which 7 will be seen only if the eye be more than 20° from the bright point; and in other positions of the eye 6 only are visible.

8. An eye views itself by repeated reflections in two plane mirrors inclined at any angle. Prove that if it look directly at any image formed by an odd number of reflections, the image of the pupil will appear circular.

9. Prove that, if γ be the angle between two plane mirrors and n the nearest odd integer to $2\pi/\gamma$, then the number of images formed is n if the luminous object be more than $\frac{1}{2}n\gamma \sim \pi$ from the bisector of the angle; and is $n-1$ if the distance be less than $\frac{1}{2}n\gamma - \pi$, or $n+1$ if the distance be less than $\pi - \frac{1}{2}n\gamma$, according as the one or other of these quantities be positive.

10. Two vertical plane mirrors make an angle γ with each other and a person stands between them.

Prove that (i) when γ lies between $\pi/2n$ and $\pi/(2n+1)$ he can see $(4n+1)$ or $4n$ images of himself as his angular distance from the bisector of the angle is greater or less than $\{(2n+1)\gamma - \pi\}/2$;

(ii) when γ lies between $\pi/(2n+1)$ and $\pi/(2n+2)$ he can see $(4n+1)$ or $(4n+2)$ images of himself as his angular distance from the bisector is greater or less than $\{\pi - (2n+1)\gamma\}/2$.

11. Three plane mirrors are placed so that their intersections are parallel to each other, and the section made by a plane perpendicular to them is an acute-angled triangle.

A ray of light after one reflection in this plane at each mirror is parallel to its original direction; shew that after another reflection at each mirror it will pursue its original path, and that the whole length of the path from any point of it to that point again is twice the perimeter of the pedal triangle.

12. Four plane rectangular mirrors $AA'B'B$, $BB'C'C$, $CC'D'D$, $DD'A'A$ of equal lengths AA', BB', CC', DD', but of different breadths, are enclosed in a right circular cylinder, the common edges AA', BB', CC', DD' being in contact with its inner surface; prove that the direction of a ray of light is not altered by four successive reflections at the mirrors AB', BC', CD', DA', in this order; and that as long as the ray continues to be reflected from the mirrors in this order the points of incidence on any mirror lie at equal distances along a straight line.

13. On the sill of a window facing south is placed a small plane horizontal mirror. Shew that the curves traced on the ceiling day by day by the reflected sunlight are hyperbolas for any place south of latitude $66\frac{1}{2}°$ N., which become a straight line at the equinoxes.

14. Shew that if the altitude of the sun be θ and waves make all angles up to a small angle a with the horizon, then to an observer on a cliff the bright patch formed by reflection of the sun's rays in the sea will subtend an elliptic cone whose principal angular radii are $2a$ and $2a \sin\theta$.

15. Three plane mirrors are placed with their planes at right angles to one another. Shew that a ray reflected at them in succession is reversed in direction; and that whatever the number of reflections, every portion of the reflected ray is parallel to one of four straight lines.

16. A ray is reflected at three plane mirrors successively, so as to be parallel to its original direction, and the three directions which it takes are mutually at right angles. Prove that the mirrors are mutually inclined at angles $\frac{1}{3}\pi$.

17. A ray of light is reflected at each of three plane mirrors which are all parallel to one straight line, and the final direction of the ray is parallel to its direction before incidence. Shew that the incident ray must be parallel to one of three fixed planes, and the portions after the first and second reflections parallel to the other two; and find the directions of these planes.

18. A pencil of rays from any point on the focal ellipse of an ellipsoid is reflected at the surface; shew that all the reflected rays intersect the focal ellipse.

19. A beam of light parallel to the axis is reflected at an elliptic paraboloid whose equation is $by^2+cz^2=2x$.

Shew that the equations of the reflected ray at the point (f, g, h) are

$$\frac{x-f}{b^2g^2+c^2h^2-1}=\frac{y-g}{2bg}=\frac{z-h}{2ch};$$

and that all the reflected rays pass through two parabolas lying in the principal planes of the paraboloid.

CHAPTER II.

REFRACTION.

14. WHEN a ray of light passes from one homogeneous medium into another it undergoes a change of direction, and is said to be *refracted*. The acute angles made by the two parts of the ray with the normal to the surface of separation of the two media at the point of incidence are called the angles of incidence and refraction.

The complete relation between the two directions is given by the following laws.

(i) *The two parts of the ray are in the same plane with the normal and on opposite sides of it.*

(ii) *The ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant, depending on the media, and on the nature of the light.*

The relation between the sines was discovered by Willebrod Snell, d. 1626; that the ratio varies with the nature of the light was proved by Newton. This latter property, which is the cause of chromatism, is dealt with in Chap. VII. on Achromatism; for the present we shall regard the ratio as known and constant.

This ratio is called the *index of refraction* from the first medium to the second; when the ray passes from any medium into a denser medium, it is bent nearer the normal; and in this case the relative index of refraction is greater than unity.

The *absolute index of refraction* of a medium is the index of refraction from a vacuum into that medium; and we proceed to shew that the index of refraction from a medium A into a medium B is the absolute index of the medium B divided by that of A.

It is a fact, capable of direct experimental verification, that if a ray pass through any number of media separated by parallel planes, and if the final medium be the same as the initial medium, the final direction of the ray is always parallel to its initial direction.

First, let a ray pass from one medium A through a medium B bounded by parallel planes, into the medium A again, and let ${}_a\mu_b$ denote the index of refraction from A into B, ${}_b\mu_a$ that from B into A.

Then if ϕ and ϕ' be the angles of incidence and refraction at the first refraction, the angle of incidence will be ϕ' at the second refraction, and the angle of refraction there will be found to be ϕ. Hence by Snell's law,

$$\sin\phi = {}_a\mu_b \sin\phi'; \quad \sin\phi' = {}_b\mu_a \sin\phi.$$

Therefore $${}_a\mu_b \cdot {}_b\mu_a = 1.$$

Secondly, let the ray pass from the medium A through two media B, C, bounded by parallel planes, and emerge into A; then if ϕ, ϕ', ϕ'' be the successive angles of incidence, ϕ will be the angle of refraction at the last surface.

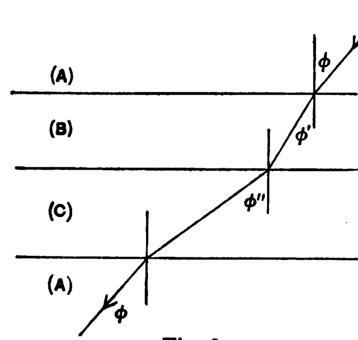


Fig. 9.

Also

$$\sin\phi = {}_a\mu_b \sin\phi',$$
$$\sin\phi' = {}_b\mu_c \sin\phi'',$$
$$\sin\phi'' = {}_c\mu_a \sin\phi.$$

Hence

$${}_a\mu_b \cdot {}_b\mu_c \cdot {}_c\mu_a = 1,$$

and therefore $${}_a\mu_b = {}_c\mu_b / {}_c\mu_a.$$

If the medium C be a vacuum, then ${}_c\mu_a$ is the absolute refractive index of the medium A; and we may therefore write Snell's law in the form

$$\mu \sin\phi = \mu' \sin\phi',$$

where μ and μ' are the absolute indices of the media in which the angles ϕ and ϕ' are measured.

The path of a ray through any system of media bounded by

parallel planes may be found from the equation $\mu \sin \phi =$ constant, ϕ being the angle of incidence in the medium of absolute refractive index μ.

15. Critical angle.

If the medium in which the ray is incident be denser than the medium into which it is refracted, the ray is bent away from the normal, the angle of refraction ϕ' being given in terms of the angle of incidence ϕ by the equation $\sin \phi' = \frac{\mu}{\mu'} \sin \phi$, where $\mu > \mu'$.

We see that ϕ' increases with ϕ, and that ϕ' reaches its greatest possible value $\frac{1}{2}\pi$, when ϕ is $\sin^{-1}(\mu'/\mu)$.

This angle is called the *critical angle* for the two media, and the ray then emerges tangentially to the surface of separation.

For values of ϕ greater than this value, the equation above gives an impossible value to $\sin \phi'$, and it is found that light incident at an angle greater than the critical angle is reflected at the surface in accordance with the ordinary law.

When a ray of light is incident at the surface of separation of two transparent media, part of the light is reflected and part refracted, in proportions varying with the angle of incidence; but in the case of incidence at an angle greater than the critical angle, there is no refracted ray, and all the incident light is reflected. Hence reflection in this manner is known as *total internal reflection.*

The numerical values of the absolute refractive indices of some of the principal media are

Air at 0° C. and 760 mm. barometer	1·000294
Pure water	1·336
Crown-glass	1·531 to 1·563
Flint-glass	1·576 to 1·642
Diamond	2·44 to 2·75
Chromate of lead	2·50 to 2·97

The critical angle for refraction from water to air is 48° 27′, and this also is the extreme angle which can be made with the normal by any ray entering into water from air, so that an eye below the surface of still water will therefore see all external objects apparently comprised within a cone of that semi-vertical

angle. Those objects that are at any distance from the normal through the eye will necessarily appear much distorted.

16. *The deviation at a single refraction increases with the angle of incidence, or of refraction, at an increasing rate.*

For a ray refracted into a denser medium, we have, if μ be the relative refractive index, $\sin\phi = \mu \sin\phi'$, where $\mu > 1$, and $\phi > \phi'$.

The deviation D is therefore $\phi - \phi'$. Differentiating the relation between the sines, we have

$$\cos\phi d\phi = \mu \cos\phi' d\phi'.$$

Hence
$$\left(\frac{d\phi}{d\phi'}\right)^2 = \frac{\mu^2 \cos^2\phi'}{\cos^2\phi} = \frac{\mu^2 - \sin^2\phi}{\cos^2\phi}$$
$$= (\mu^2 - 1)\sec^2\phi + 1.$$

Therefore $\dfrac{d\phi}{d\phi'}$ is greater than unity, and moreover increases with ϕ, since $\sec\phi$ increases with ϕ.

Hence ϕ increases faster than ϕ', and therefore D increases with ϕ or ϕ', and moreover at an increasing rate.

(In fact the first three differential coefficients of D with regard to ϕ are necessarily positive.)

When the ray is incident in the denser on the rarer medium, the deviation is numerically the same as in the previous case, for the path of a ray of light can be reversed.

We may also prove the theorem of this article by a geometrical construction due to Tait.

Let C be the centre of a circle of radius a, and O a point such that $CO = \mu a$.

If a vector OP be drawn so that the angle COP is ϕ', then the angle external to the angle CPO will be ϕ; for

$\sin CPO/\sin COP = CO/CP = \mu.$

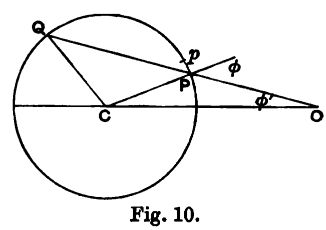


Fig. 10.

The deviation is $\phi - \phi'$, and is represented by the angle PCO. Hence as P moves round the circle from the line CO till OP touches the circle, ϕ, ϕ' and $\phi - \phi'$ will increase together.

Moreover, let P move to an adjacent point p, then the arc Pp, which is $a\partial(\phi-\phi')$, is also equal to $OP\partial\phi'\sec\phi$.

Hence $$\frac{d(\phi-\phi')}{d\phi'}=\frac{OP}{a}\sec\phi;$$

and both factors on the right-hand side increase (at increasing rates) as ϕ' and ϕ increase.

COROLLARY. To prove that

$\sin(\phi-\phi')/\sin\phi'$, and $\sin(\phi-\phi')/\sin(\phi+\phi')$ increase with ϕ.

(i) Plainly $\sin(\phi-\phi')/\sin\phi'=OP/a$,

and OP increases as ϕ increases.

(ii) Let OP cut the circle again in Q, then the angle OCQ is supplementary to $\phi+\phi'$, and therefore

$$\sin(\phi-\phi')/\sin(\phi+\phi')=OP/OQ;$$

where the numerator increases and the denominator decreases as ϕ increases.

17. Examples.

1. *A refracting semicircular cylinder of radius a is laid with its plane face on a horizontal table. Light moving in a horizontal direction perpendicular to the axis falls on the cylinder. Shew that if $\mu>2$ all the rays will be totally reflected at points within the base of the cylinder; and that if a vertical screen be placed above the highest generator of the cylinder, the back of this screen beyond a distance $\frac{\mu^2}{\mu^2-2}a$ from the axis will be illuminated.*

Let a ray incident at P at angle ϕ be refracted in the direction PQ; then

$$OQ/a=\sin\phi'/\sin(\phi-\phi'),$$

which is least when ϕ is $\frac{1}{2}\pi$, and greatest when ϕ is zero. Its limiting value in the latter case is $1/(\mu-1)$. Hence all the rays after refraction are incident at points within the base, if $\mu>2$. Moreover the angle of incidence at Q is $\frac{1}{2}\pi-(\phi-\phi')$, which is least when ϕ is $\frac{1}{2}\pi$, and is then equal to the critical angle. Hence all the rays must be totally reflected at the base.

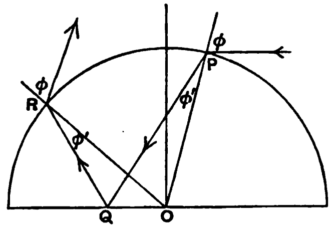


Fig. 11.

Again, if the reflected ray be incident on the curved surface at R, then

$$\sin ORQ/\sin OQR=OQ/OR,$$

and the angles OQR, OQP are supplementary. Hence the angle ORQ is ϕ', and the angle of emergence is ϕ.

The angle that QR makes with the base is $\phi-\phi'$, the deviation at R is also $\phi-\phi'$; hence the emergent ray will meet the back of the screen if

$$2(\phi-\phi')>\tfrac{1}{2}\pi, \text{ i.e. if } \phi-\phi'>\tfrac{1}{4}\pi.$$

The maximum value of $\phi-\phi'$ occurs when ϕ is $\frac{1}{2}\pi$, and is $\frac{1}{2}\pi-a$, where a is the critical angle. Hence this condition requires that $\frac{1}{4}\pi>a$, or $1/\sqrt{2}>1/\mu$, which is satisfied since $\mu>2$.

Also for a ray which meets the back of the screen, we easily find that the height of the point on the screen above O is $-a\sin\phi/\cos 2(\phi-\phi')$, *i.e.* is

$$a \Bigg/ \left\{2\frac{\sin(\phi-\phi')}{\sin\phi}\sin(\phi-\phi')-\operatorname{cosec}\phi\right\}.$$

As ϕ increases from the value for which $\phi-\phi'=\frac{1}{4}\pi$ to $\phi=\frac{1}{2}\pi$, then $\sin(\phi-\phi')/\sin\phi$ and $\sin(\phi-\phi')$ are increasing, $\operatorname{cosec}\phi$ is decreasing. The lowest point illuminated is therefore reached by the ray which emerges touching the circle; and its height above O is $a/(2\sin^2 a-1)$, *i.e.* $\mu^2 a/(\mu^2-2)$.

2. Any number of parallel plates of thicknesses $t_1, t_2 \dots t_n$, composed of media of which the absolute refractive indices are $\mu_1, \mu_2 \dots \mu_n$ respectively, are placed in contact. A ray is incident in a vacuum on the first; shew that the greatest possible length of the path of the ray within the plates is

$$\sum_{r=1}^{r=n} \cdot \frac{\mu_r t_r}{(\mu_r^2-1)^{\frac{1}{2}}}.$$

3. Parallel rays are incident on the whole of the curved surface of a refracting hemisphere in the direction perpendicular to its base; prove that if $\mu>\sqrt{2}$ the area of the circle on the base through which the light emerges is to the area of the base as $1:\mu^4-1-2(\mu^2-1)^{\frac{3}{2}}$; and that the ratio of the amount of the emergent light to that of the incident light is

$$1:\mu^2+1-2(\mu^2-1)^{\frac{1}{2}}.$$

4. A ball of glass contains a concentric spherical cavity; shew that, provided the radius of the cavity do not exceed the radius of the ball divided by the index of refraction μ of the glass, it will appear to an eye at any distance from the ball to be μ times greater than it really is.

5. A bright point is placed at the centre of a spherical cavity within a sphere of glass of index μ and radius a. If the distance of the point from the centre of the sphere be greater than a/μ, the bright area on a screen placed parallel to the lines of centres will be traversed by a dark band bounded by two hyperbolas.

6. A cylindrical pencil of light is incident on a refracting prolate spheroid in a direction parallel to the axis, the eccentricity of the spheroid being e and the index of refraction being μ; find the incident rays which emerge parallel to the axis if $\mu>1/e^2$.

7. Shew that if a cylindrical pencil of rays be incident on a refracting prolate spheroid of axis $2a$ and eccentricity e, whose index of refraction is $1/e$, in a direction parallel to the axis, the emergent pencil will meet the further directrix plane in a circle of radius $a(1-e^2)^{\frac{3}{2}}(5e^2+3)/(1+e^2)(5e^2-1)$, supposing that $e^2>1/5$.

18. *When a ray is refracted at a plane, the angles which the incident and refracted rays make with any plane through the normal, obey the law of refraction; and the projections of the rays on the plane through the normal obey a modified law of refraction.*

Let PO, OQ be the incident and refracted rays in media of indices μ and μ' respectively.

Take PO to OQ in the ratio $\mu : \mu'$, and draw PM, QN perpendicular to the normal to the refracting surface. Then since

$$\mu \sin \phi = \mu' \sin \phi',$$

PM and NQ are equal and parallel.

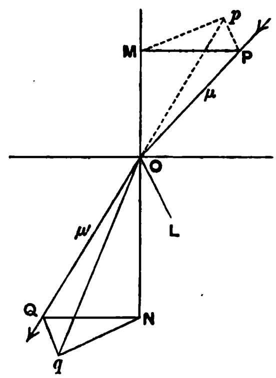


Fig. 12.

Hence their projections on any direction are equal; or the sum of the projections of PO, OM = sum of the projections of NO, OQ.

Let OL be any direction in the refracting plane, and let the angle between PO and OL be θ, and between OQ and OL be θ'. Then since MON is perpendicular to OL, we have, by projecting on OL,

$$\mu \cos \theta = \mu' \cos \theta' \quad\ldots\ldots\ldots\ldots(1).$$

If OL be normal to a plane through MON, and Op, Oq be the projections of OP, OQ on this plane, then the angle made by PO with the plane is POp, and is complementary to θ. Denoting the angles made by PO, OQ with the plane through the normal by α, α', we have

$$\mu \sin \alpha = \mu' \sin \alpha' \quad\ldots\ldots\ldots\ldots(2).$$

Again, the projections pM, Nq of the equal and parallel lines PM, NQ are themselves equal and parallel; hence if the projections pO, Oq of the ray make angles χ, χ' with the normal, we have

$$Op \sin \chi = Oq \sin \chi'$$

or

$$\mu \cos \alpha \sin \chi = \mu' \cos \alpha' \sin \chi' \quad\ldots\ldots\ldots\ldots(3).$$

19. Direction-cosines of the refracted ray.

Let (l, m, n) be the direction-cosines, referred to any rectangular axes, of the incident ray PO; (l', m', n') those of the

refracted ray OQ, and let (p, q, r) be the direction-cosines of the normal MN, taken in the direction from the medium in which the light is incident into the medium in which it is refracted.

Then since the projections of PO, OM on any axis = the projections of NO, OQ on that axis, we have

$$\mu l - \mu \cos\phi \,.\, p = -\mu' \cos\phi' \,.\, p + \mu' l',$$

and two similar equations.

Hence

$$(\mu' l' - \mu l)/p = (\mu' m' - \mu m)/q = (\mu' n' - \mu n)/r = \mu' \cos\phi' - \mu \cos\phi.$$

Also $\cos\phi = lp + mq + nr$, and ϕ' is given by the equation $\mu \sin\phi = \mu' \sin\phi'$.

If the values chosen for p, q, r should happen to give $\cos\phi$ negative, then $\cos\phi'$ must also be taken negative.

20. Prism.

A prism is a portion of some homogeneous refracting medium, bounded by two planes, which meet in the edge of the prism. The angle of the prism is the angle between these planes; a plane perpendicular to the edge is called a principal plane of the prism. We confine ourselves at present to rays passing through the prism in a principal plane.

Denote the angle of the prism by i, the angles of incidence and refraction at the first face by ϕ and ϕ', and of incidence and

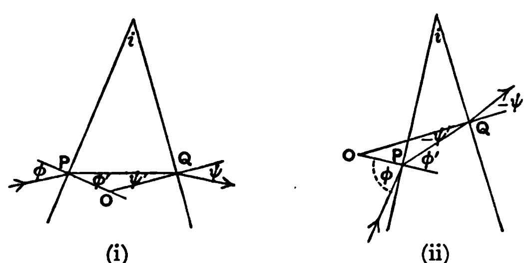


Fig. 13.

emergence at the second face by ψ' and ψ. Then ϕ and ϕ' necessarily lie on opposite sides of the normal to the first face, similarly ψ and ψ' are on opposite sides of their normal; if the convention be adopted that ϕ and ψ are reckoned positive when measured from the normals away from the edge, ϕ' and ψ' will be positive when measured from the normals towards the edge.

Under this convention we have in all cases the equations

$$\sin\phi = \mu\sin\phi', \quad \sin\psi = \mu\sin\psi', \quad \phi' + \psi' = i, \quad D = \phi + \psi - i,$$

D being the deviation of the ray.

In the case where the angles involved are all positive, they are obviously true; if two of the angles ψ and ψ' be negative, as in the figure (ii), then, the normals at P and Q meeting at O, the angle $POQ = i = \phi' - (-\psi') = \phi' + \psi'$; and the deviation at P is $\phi - \phi'$ away from the edge of the prism, and at Q is $-\psi - (-\psi')$ towards the edge. Therefore $D = \phi - \phi' + \psi - \psi' = \phi + \psi - i$.

The case of ϕ and ϕ' being negative may be included in this one by supposing the path of the ray reversed, and exchanging ϕ and ψ, ϕ' and ψ'.

21. *In any prism denser than the surrounding medium the deviation is always from the edge.*

This is obvious when the angles are all positive, for the deviations both at P and Q are from the edge. But if ϕ be positive and ψ negative, in which case the deviations at P and Q are in opposite directions, then since ϕ' is greater than the numerical value of ψ', the deviation at P (cf. § 16) is greater than the deviation at Q, and therefore on the whole the deviation is from the edge.

Similarly when ϕ' is negative, ψ' is positive and greater than the numerical value of ϕ', and the deviation at emergence is greater than that on entering the prism, so that the whole deviation is from the edge.

If the prism be less refractive than the surrounding medium, the deviation is equal to $\phi' + \psi' - \phi - \psi$, and is always towards the edge.

22. *The deviation is a minimum when the ray passes symmetrically.*

Consider a ray passing symmetrically, let ϕ' increase by any quantity, then ψ' must decrease by an equal amount.

Hence the deviation at the first face must increase, while that at the second decreases. But since $\phi' > \psi'$ the rate of increase is greater than the rate of decrease, and this inequality will hold till ϕ and ϕ' attain their maximum values, viz. $\frac{1}{2}\pi$ and the critical angle.

Hence the deviation on the whole increases.

Similarly if ϕ' decrease and ψ' increase from equality, the deviation increases.

The deviation is therefore a minimum when the ray passes symmetrically, and in this case $\phi' = \psi' = \frac{1}{2}i$, $\phi = \psi = \frac{1}{2}(D+i)$; so that the minimum deviation is given by $\sin\frac{1}{2}(D+i) = \mu\sin\frac{1}{2}i$.

This theorem can also be proved thus. We have

$$\sin\phi + \sin\psi = \mu(\sin\phi' + \sin\psi'),$$

i.e.
$$\sin\frac{\phi+\psi}{2}\cos\frac{\phi-\psi}{2} = \mu\sin\frac{\phi'+\psi'}{2}\cos\frac{\phi'-\psi'}{2},$$

or
$$\sin\tfrac{1}{2}(D+i) = \mu\sin\tfrac{1}{2}i\,\frac{\cos\frac{1}{2}(\phi'-\psi')}{\cos\frac{1}{2}(\phi-\psi)}.$$

Now if $\phi > \psi$, $\phi - \phi' > \psi - \psi'$ or $\phi - \psi > \phi' - \psi'$,

and if $\psi > \phi$, $\psi - \psi' > \phi - \phi'$ or $\psi - \phi > \psi' - \phi'$.

In either case $\cos\frac{1}{2}(\phi \sim \psi) < \cos\frac{1}{2}(\phi' \sim \psi')$, since these are always acute angles, and therefore the least value of $\sin\frac{1}{2}(D+i)$ is $\mu\sin\frac{1}{2}i$, when $\phi = \psi$. It follows, since $\frac{1}{2}(D+i)$ is an acute angle, that D is least for the symmetrical path, and it has been shewn above that D is greatest for the path in which either ϕ or ψ is $\frac{1}{2}\pi$.

23. Limitation of the field of view as seen through a prism.

Not every ray that falls on the first face can emerge at the second face; in order that this may be possible the numerical value of the angle of incidence there must be less than a, the critical angle for the material of the prism.

Since $\phi' + \psi' = i$, no ray can possibly emerge if $i > 2a$, since ϕ' and ψ' both attain their greatest possible value when equal to a.

If $i < 2a$, then the greatest value of ψ' for emergence being a, the least value of ϕ' is $i - a$ and the least value of ϕ is $\sin^{-1}\{\mu\sin(i-a)\}$. Similarly the greatest value of ϕ' being a, the least value of ψ' is $i-a$, and of ψ is $\sin^{-1}\{\mu\sin(i-a)\}$.

The angular breadth of the field of view is therefore $\frac{1}{2}\pi - \sin^{-1}\{\mu\sin(i-a)\}$, both in reality and in appearance.

The deviation has its greatest value both when ϕ is greatest and when ψ is greatest; the edges of the field of view are therefore turned through the same angle, but since the deviation changes most rapidly near this extreme value the field is most distorted there.

When the deviation is a minimum, its value is stationary, *i.e.* the angle between two near emergent rays is equal to that between the incident rays; and therefore a small pencil, of which the axis passes with minimum deviation, gives the best image.

24. The image of a small object, viewed through a prism so that a mean ray passes near the edge with minimum deviation, is equal to the object.

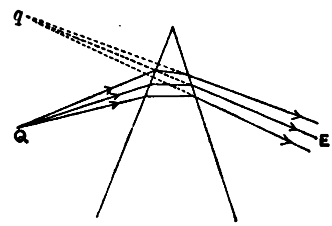


Fig. 14.

If Q, a point of the object, be the origin of a small pencil the axis of which passes symmetrically, then the deviation is *stationary* and therefore very nearly the same for all rays of the pencil. The divergence of the pencil will therefore be unaltered.

The incident and emergent mean rays are symmetrical with respect to the plane bisecting the angle of the prism; hence consecutive emergent rays, if produced backwards, will intersect in a point q, at the same distance as Q from this plane.

If the focal centre E of the eye lie on the emergent mean ray, we see by retracing the path of a small pencil which enters E, that any small object at Q appears to lie at q, and that each part of it subtends the same angle at the eye as if viewed directly.

25. Example.

A ray is incident in a principal plane at the base of a triangular prism and after being reflected at the other two faces emerges at the base. Shew that, when the angle of incidence is equal to the angle of emergence measured in opposite directions from the normal, the deviation is a maximum.

Adopting the convention of signs of Art. 20, let ϕ be the angle of incidence of the ray, ϕ' the angle of refraction, θ_1 and θ_2 the angles of reflection at the sides, and ψ' and ψ the angles of incidence and emergence at the base of the prism ABC.

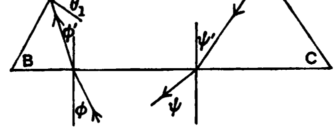


Fig. 15.

Then

$$\phi'+\theta_1=B, \quad \theta_1+\theta_2=A, \quad \theta_2+\psi'=C \ldots(1),$$

and the deviation measured in the direction BAC is

$$(\phi-\phi')+(\pi-2\theta_1)+(\pi-2\theta_2)+(\psi-\psi') \,..\, (2).$$

Hence

$$\phi'+\psi'=B+C-A=\pi-2A$$

and
$$D=\pi+\phi+\psi.$$

If A be an acute angle, $\phi'+\psi'$ must be positive, and therefore either ϕ' and ψ' are both positive, or the negative angle is numerically the smaller. Hence either ϕ and ψ are both positive, or the negative one is numerically the smaller.

It follows that the deviation as measured is greater than two right angles; so that, defining the deviation as the angle less than two right angles in either direction through which the ray is bent, we have

$$D=2\pi-(\pi+\phi+\psi)=\pi-\phi-\psi.$$

Hence, as in Art. 23,

$$\cos\frac{D}{2}=\sin\frac{\phi+\psi}{2}=\mu\sin\frac{\phi'+\psi'}{2}\cos\frac{\phi'-\psi'}{2}\Big/\cos\frac{\phi-\psi}{2}$$

$$=\mu\cos A\cos\frac{\phi'-\psi'}{2}\Big/\cos\frac{\phi-\psi}{2},$$

and the ratio of the two cosines of the differences being never less than unity (cf. § 22), it follows that $\cos\frac{1}{2}D$ is least and therefore D is a maximum, when $\phi=\psi$.

But if A be an obtuse angle, then $\phi'+\psi'$ must be negative, and therefore $\phi+\psi$ must be negative, and $D\,(=\pi+\phi+\psi)$, is less than π.

Hence in this case

$$\cos\tfrac{1}{2}D=-\mu\cos A\cos\frac{\phi'-\psi'}{2}\Big/\cos\frac{\phi-\psi}{2},$$

and we obtain as before that D is a maximum when $\phi=\psi$.

The deviation in the first case will be least when ϕ or ψ attains its greatest possible value $+\frac{1}{2}\pi$; and in the second case when either attains the value $-\frac{1}{2}\pi$.

26. Example.

Prove that the deviation in a prism increases at an increasing rate as the angle of incidence increases from its value for minimum deviation.

Hence shew that if two equal prisms be placed with their edges coincident and the faces making a given angle with each other, the deviation of a ray passing through both is a minimum for the symmetrical path.

With the usual notation $D=\phi+\psi-i$; $\phi'+\psi'=i$. Also on differentiating the relation $\sin\phi=\mu\sin\phi'$ logarithmically, $d\phi/\tan\phi=d\phi'/\tan\phi'$.

Hence $$\frac{dD}{d\phi'}=\frac{\tan\phi}{\tan\phi'}-\frac{\tan\psi}{\tan\psi'};$$

and if $\phi>\psi$, this rate of increase is known to be positive (cf. § 22).

Again, $$\frac{d}{d\phi'}(\tan\phi/\tan\phi')=(\sec^2\phi\tan\phi-\tan\phi\sec^2\phi')/\tan^2\phi'$$
$$=\tan\phi\,(\cos^2\phi'-\cos^2\phi)/\cos^2\phi\sin^2\phi'$$
$$=(\mu^2-1)\sin\phi\sec^3\phi.$$

Hence $$\frac{d^2D}{d\phi'^2}=(\mu^2-1)(\sin\phi\sec^3\phi+\sin\psi\sec^3\psi),$$

which is positive even if ψ be negative, since $\phi>$ the numerical value of ψ. The rate of increase of the deviation in a prism therefore increases with the angle of refraction, and with the angle of incidence.

When two equal prisms of angle i have their edges coincident and the adjacent faces inclined at angle β, then a ray can take the symmetrical path only if i be less than the critical angle a, and if further

$$\sin \tfrac{1}{2}\beta < \sin (a - i)/\sin a,$$

as shewn in Art. 23.

But if these conditions be satisfied, then the symmetrical path makes the total deviation a true minimum.

For any change from this path will increase the angle of incidence on the second prism and decrease the angle of emergence from the first by the same amount, or conversely, will decrease the first angle and increase the second.

The rate of increase of the deviation in the one prism will always be greater than the rate of decrease in the other. Hence the total deviation must increase as we recede from the symmetrical path in either direction.

27. Application of Spherical Trigonometry.

The representation of the angles between the various parts of the path of a ray, which do not necessarily lie in the same plane, is facilitated by making use of spherical trigonometry, and drawing radii of a sphere parallel to the successive directions.

Let radii of a sphere be drawn parallel to the incident and refracted rays and the normal, meeting the sphere in the points P, Q, N respectively; then PQN is a great circle and

$$\mu \sin NP = \mu' \sin NQ.$$

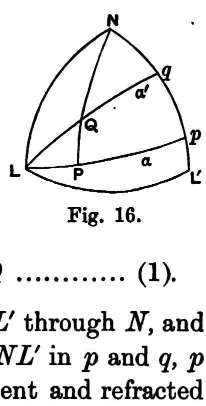


Fig. 16.

If L represent any line in the refracting plane $LN = \tfrac{1}{2}\pi$, and therefore

$$\mu \cos LP = \mu \sin NP \cos LNP$$

$$= \mu' \sin NQ \cos LNQ = \mu' \cos LQ \quad \text{............} \quad (1).$$

Again, if L be the pole of any great circle NL' through N, and if the great circles LPp, LQq be drawn to meet NL' in p and q, p and q will represent the projections of the incident and refracted rays on the plane through the normal, and if $Pp = \alpha$, $Qq = \alpha'$, then

$$\mu \sin \alpha = \mu' \sin \alpha' \quad \text{.....................} \quad (2).$$

Let the angle PNp be ω, and let $Np = \chi$, $Nq = \chi'$; then from the right-angled spherical triangle NpP, we have

$$\tan \chi = \cos \omega \tan \phi.$$

Hence

$$\tan \chi / \tan \chi' = \tan \phi / \tan \phi' \quad \text{..............} \quad (3).$$

Again, $$\cot \omega = \cot \alpha \sin \chi = \cot \alpha' \sin \chi'.$$

Hence $$\mu \cos \alpha \sin \chi = \mu' \cos \alpha' \sin \chi' \qquad (4).$$

The above is an alternative proof of the theorems of Art. 18.

28. Path of a ray through a prism, not in a principal plane.

Let radii of a sphere drawn parallel to the directions of the ray before incidence, in the prism, and on emergence, meet the sphere in P, Q, R respectively; and let radii parallel to the normals to the faces meet the sphere in N_1 and N_2. Then, if the first normal be drawn onwards into the prism, the second should be drawn onwards and therefore out of the prism, so that the arc N_1N_2 is i, the angle of the prism.

The pole E of the great circle N_1N_2 will represent the edge of the prism, as the great circle itself represents the principal plane.

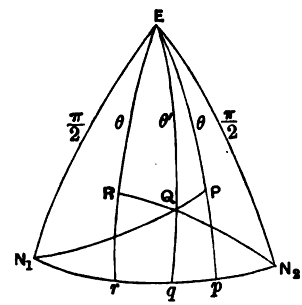


Fig. 17.

Let ϕ and ϕ' be the angles of incidence and refraction at the first face, ψ' and ψ those at the second face. Then $N_1P = \phi$, $N_1Q = \phi'$, $N_2Q = \psi'$, $N_2R = \psi$.

If the angle PN_1N_2 between the plane of incidence and the principal plane be given, then the angle RN_2N_1 between the plane of emergence and the principal plane can be found as follows:—denoting these angles by ω_1 and ω_2 we have

$$\sin \phi = \mu \sin \phi',$$

$$\cos \psi' = \cos \phi' \cos i + \sin \phi' \sin i \cos \omega_1,$$

$$\cot \omega_2 = -\cos i \cot \omega_1 + \sin i \cot \phi' \operatorname{cosec} \omega_1,$$

$$\sin \psi = \mu \sin \psi'.$$

Also if θ be the angle made by the incident ray with the edge of the prism, and θ' the angle made by the ray in the prism, we have $\cos \theta = \mu \cos \theta'$; hence the emergent ray makes the same angle θ with the edge, since the edge is a line lying in both the refracting planes.

The deviation of the ray on passing through the prism is

measured by the arc PR, and from the isosceles triangle PER the deviation is given by

$$\sin \tfrac{1}{2}D = \sin\theta \sin \tfrac{1}{2}(PER).$$

The possible angles of incidence and emergence can be constructed graphically. With N_1 and N_2 as poles describe small circles of radius equal to the critical angle; the point Q, which represents the direction of the ray in the prism, must lie within the area of the sphere common to these two circles, and the corresponding positions of P and R can be determined.

29. The direction of the emergent ray may be obtained in a simpler manner than the above by using the projections of the ray on the principal plane.

Let p be the projection of P on the great circle N_1N_2, and let $N_1p = \chi_1$, $Pp = \alpha$; then if ϕ and ω_1 be given, α and χ_1 are determined from the equations

$$\sin\alpha = \sin\phi \sin\omega_1, \quad \tan\chi_1 = \cos\omega_1 \tan\phi.$$

Again, let q be the projection of Q, and let $N_1q = \chi_1'$, $N_2q = \chi_2'$, $Qq = \alpha'$; then

$$\sin\alpha = \mu \sin\alpha',$$

$$\cos\alpha \sin\chi_1 = \mu \cos\alpha' \sin\chi_1',$$

$$\chi_1' + \chi_2' = i.$$

Lastly, if r be the projection of R, and $N_2r = \chi_2$, $Rr = \alpha$, we have

$$\cos\alpha \sin\chi_2 = \mu \cos\alpha' \sin\chi_2'.$$

And ψ and ω_2 are determined from the equations

$$\cos\psi = \cos\alpha \cos\chi_2, \quad \cot\omega_2 = \cot\alpha \sin\chi_2.$$

Here the signs to be attached to χ_1, χ_2, and χ_1', χ_2' are determined by the same rules as those of the external angles ϕ, ψ, and the internal angles ϕ', ψ', when the path is in the principal plane (cf. § 20).

Also $$\sin\tfrac{1}{2}D = \cos\alpha \sin\tfrac{1}{2}(\chi_1 + \chi_2 - i).$$

30. *The deviation is a minimum when the ray passes symmetrically in a principal plane.*

The deviation is a function of two independent variables, such as the angle of incidence and the azimuth of the plane of incidence, or of any two combinations of these variables. We shall

shew however that, if a ray be incident in a direction making a given angle with the edge, the deviation is least when the path is symmetrical, and next, that of these minima, the path in the principal plane affords the least.

Since the angle α is given, so also is α', for $\sin\alpha = \mu \sin\alpha'$; these angles are acute, and since $\mu > 1$, $\alpha > \alpha'$, so that

$$\mu \cos\alpha' > \cos\alpha.$$

Also with the notation of Art. 29 we have

$$\cos\alpha \sin\chi_1 = \mu \cos\alpha' \sin\chi_1'.$$

Hence we deduce as in Art. 16 that $d\chi_1/d\chi_1'$ is greater than unity, and increases as χ_1' increases.

Beginning then with the symmetrical path in which q bisects N_1N_2, and the path of the ray in the prism is parallel to the plane bisecting the external angle, we see that χ_1' must increase and χ_2' decrease at the same rate, since $\chi_1' + \chi_2' = i$.

Hence χ_1 increases faster than χ_2 decreases. But

$$\sin \tfrac{1}{2}D = \cos\alpha \sin \tfrac{1}{2}(\chi_1 + \chi_2 - i);$$

and hence, for a given value of α, D will increase as we pass from the symmetrical path.

Again, in the symmetrical path, we have

$$\sin \tfrac{1}{2}D = \sin PQ \sin PQE = \sin PQ \sin \tfrac{1}{2}i/\sin\phi';$$

or $$\sin \tfrac{1}{2}D/\sin \tfrac{1}{2}i = \sin(\phi - \phi')/\sin\phi'.$$

But as the angle N_2N_1Q increases from zero, ϕ' must increase, and therefore $\sin(\phi - \phi')/\sin\phi'$ increases (cf. Corollary, Art. 16).

Hence the symmetrical path in the principal plane gives absolutely minimum deviation.

31. Example. *To find the greatest deviation produced by a prism*.*

With the notation of Art. 28, we have from the spherical triangles PQR and N_1QN_2,

$$\cos D = \cos(\phi - \phi')\cos(\psi - \psi') + \sin(\phi - \phi')\sin(\psi - \psi')\cos PQR,$$
$$\cos i = \cos\phi' \cos\psi' + \sin\phi' \sin\psi' \cos PQR.$$

Substituting for $\cos PQR$ and reducing the terms by the use of the equations $\sin\phi = \mu\sin\phi'$, $\sin\psi = \mu\sin\psi'$, we have

$$\cos D = \mu\{\cos(\phi - \phi') + \cos(\psi - \psi')\} - \mu^2 + \frac{\sin(\phi - \phi')\sin(\psi - \psi')}{\sin\phi'\sin\psi'}\cos i.$$

* Anderson, *Camb. Phil. Soc. Proc.* Vol. IX. p. 196.

Let ϕ and therefore ϕ' be given, but let the plane of incidence be variable. Then on differentiating,

$$\sin D \frac{dD}{d\psi'} = \mu \sin(\psi - \psi')\left(\frac{d\psi}{d\psi'} - 1\right)$$
$$+ (\mu \cos\phi' - \cos\phi)\left(\mu \sin\psi' - \sin\psi \frac{d\psi}{d\psi'}\right)\cos i$$
$$= \mu \sin(\psi - \psi')\left(\mu \frac{\cos\psi'}{\cos\psi} - 1\right)$$
$$+ (\mu \cos\phi' - \cos\phi)\left(\mu \sin\psi' - \sin\psi \frac{\mu \cos\psi'}{\cos\psi}\right)\cos i$$
$$= \frac{\mu \sin(\psi - \psi')}{\cos\psi}\left\{\frac{\sin(\psi - \psi')}{\sin\psi'} - \frac{\sin(\phi - \phi')}{\sin\phi'}\cos i\right\}.$$

Now when ϕ' is given, it is plain from the figure of Art. 28 that ψ' increases as the plane of incidence recedes from the principal plane, and as ψ' increases, so also does $\sin(\psi - \psi')/\sin\psi'$ (Art. 16).

Hence if $\frac{dD}{d\psi'}$ be positive when the ray passes in the principal plane, it is always positive; and the deviation therefore, for a given value of ϕ, is greatest when ψ is $\frac{1}{2}\pi$.

The second factor of $\frac{dD}{d\psi'}$ is the possibly negative factor, and this is equal to
$$\mu \cos\psi' - \cos\psi - (\mu \cos\phi' - \cos\phi)\cos i,$$
which in the principal plane is equal to
$$\mu\{\cos(i - \phi') - \cos\phi' \cos i\} - \cos\psi + \cos\phi \cos i$$
$$= \mu \sin\phi' \sin i + \cos\phi \cos i - \cos\psi = \cos(\phi - i) - \cos\psi$$
$$= 2 \sin\tfrac{1}{2}(\phi + \psi - i) \sin\tfrac{1}{2}(\psi - \phi + i),$$
which is positive if $\phi < i + \psi$.

Again, this condition continuing to be satisfied, we know that keeping the inclination of the ray to the edge constant, the deviation can be increased by increasing ϕ. Hence the greatest possible deviation of all occurs when the ray both enters and leaves at grazing incidence; provided the condition $\frac{1}{2}\pi < i + \psi$ is now satisfied for the principal plane. But when the ray is incident in that plane at angle $\frac{1}{2}\pi$, $\sin\psi = \sin(i - a)/\sin a$, where a is the critical angle. Hence the necessary condition is that $\cos i < \sin\psi$, *i.e.*
$$\cos i \sin a < \sin i \cos a - \cos i \sin a,$$
or
$$2 \cot i < \cot a.$$
The greatest possible deviation is then given by
$$\sin\tfrac{1}{2}D_1 = \cot a \sin\tfrac{1}{2}i.$$

But if $2\cot i > \cot a$, then, taking $\phi = \frac{1}{2}\pi$, the deviation will begin to decrease as we recede from the principal plane, but at a certain value of ψ the differential coefficient must change sign, and the deviation then increases to the value D_1. This value may or may not be greater than the greatest deviation in the principal plane. For instance, if $\mu = 3/2$, $a = 41° 48'$, then taking $i = 45°$ the value of D_1 is 50° 41′, and of the greatest deviation in the principal plane 49° 48′; but if $i = 30°$ the corresponding values are 33° 40′ and 42° 8′.

EXAMPLES.

1. Three rods OA, OB, OC of lengths a, a and μa respectively ($\mu > 1$) are freely hinged at O: C is fixed and A, B slide on two fixed lines CE, CF, the angles CAO, CBO being obtuse; prove that if a ray of light incident in a direction parallel to OA pass through a prism of refractive index μ, entering it at a face perpendicular to CE and emerging at a face perpendicular to CF, its direction on emergence will be parallel to OB, whether the path of a ray lie in a principal plane or not.

2. A prism is turned round its edge. Shew that, if an object is viewed by an eye in the same principal plane with it and placed close up to the edge, the greatest angular oscillation of the image is

$$\frac{\pi}{2}+\sin^{-1}\{\mu\sin(i-a)\}-2\sin^{-1}\left(\mu\sin\frac{i}{2}\right).$$

3. Two prisms of the same angle i but of different materials are placed with a face of each in contact and their edges coincident. Shew that, if i exceed the critical angles a and β of the materials, no ray can get through; and if i be intermediate between a and β, no ray can pass if

$$\cot a+\cot\beta>2\cot i.$$

4. Prove that in a prism

$$\sin^2\tfrac{1}{2}(D+i)=\mu^2\sin^2\tfrac{1}{2}i+\sin^2\tfrac{1}{2}(\phi-\psi)-\mu^2\sin^2\tfrac{1}{2}(\phi'-\psi'),$$

and that
$$\sin^2\tfrac{1}{2}(\phi-\psi)>\mu^2\sin^2\tfrac{1}{2}(\phi'-\psi').$$

If D_0 be the minimum deviation, then

$$(\mu+1)\tan\tfrac{1}{4}D_0=\cot\tfrac{1}{2}i-(\operatorname{cosec}^2\tfrac{1}{2}i-\mu^2)^{\frac{1}{2}}.$$

5. Without knowing the angles of a triangular prism, shew that its refractive index can be determined by observing the minimum deviations of rays passing through the prism near the three angles; and if these deviations be denoted by $2a$, 2β, 2γ, then μ is given by

$$\mu^3-\mu^2(\cos a+\cos\beta+\cos\gamma)+\mu\{\cos(\beta+\gamma)+\cos(\gamma+a)+\cos(a+\beta)\}$$
$$-\cos(a+\beta+\gamma)=0.$$

6. A direct-vision spectroscope is composed of three prisms, two of which are exactly alike, and are placed each with a face in contact with the faces of the third, and their edges turned towards its blunt end. Find equations connecting the angles of the prisms and their refractive indices in order that a ray may be transmitted through the spectroscope without deviation.

If the refractive indices of the two similar prisms and the third be $\sqrt{6}$ and $\sqrt{3}+1$ respectively, and the angle of the third prism be 120°, shew that the angle of the two like prisms is $\tan^{-1}(6+3\sqrt{3})$.

7. Shew that for a ray reflected at the second face of a prism and emerging at the first, the deviation is least when the light enters or leaves at grazing incidence, and greatest when it retraces its path.

8. A ray is incident in the principal plane on the face AB of a triangular prism of cross-section ABC, and after reflection at the faces AC, CB, BA in succession emerges at the face AC. Shew that the deviation is a maximum or minimum when the light enters or leaves at grazing incidence, and distinguish between the cases.

9. The parallel faces AB, CD of a trapezium are refracting surfaces; the faces BC and AD are reflectors. A ray is incident on the side AB and after reflection at BC and AD emerges from CD. Shew that if the difference, δ, of two opposite angles of the trapezium be less than the critical angle of its material, the deviation is greatest when the ray enters or leaves at grazing incidence, and that its minimum value is $2\sin^{-1}(\mu \sin\delta)$.

10. Any number (n) of right-angled prisms are placed with their edges turned alternately in opposite directions, and a face of each in contact with a face of the next one, and a ray passes with minimum deviation.

If ϕ and ψ be the angles of incidence on the first prism and emergence from the last, shew that

$$\sin D = (\mu_1^2 + \mu_3^2 + \ldots + \mu^2_{n-1}) \sim (\mu_2^2 + \mu_4^2 + \ldots + \mu_n^2)$$

if n be even, and

$$D = 2\phi \sim \tfrac{1}{2}\pi, \quad \phi = \psi,$$

$$2\sin^2\phi = (\mu_1^2 + \mu_3^2 + \ldots + \mu_n^2) - (\mu_2^2 + \mu_4^2 + \ldots + \mu^2_{n-1})$$

if n be odd.

11. A ray is incident in a principal plane on one face of an equilateral triangular prism of side a, and after refraction into the prism is totally reflected at the second face and emerges at the third.

Prove that the greatest possible length of the path within the prism is $\sqrt{3}\mu a/2\sqrt{\mu^2-1}$ or a as μ is greater or less than 2; and that the shortest possible path is $\sqrt{3}a/2$ or $\sqrt{3}\mu a/(\sqrt{\mu^2-1}+\sqrt{3})$, as μ is greater or less than $2/\sqrt{3}$.

12. A square prism of a transparent substance, of refractive index μ ($<\sqrt{2}$), is silvered in one face, and a ray in a principal plane enters at the opposite face; shew that in order that the ray may leave the prism at the face at which it enters, after total internal reflection at each of the other faces, the point of incidence must lie within a part of the face of breadth $\{2\sqrt{\mu^2-1}-1\}$ times the side of the square.

13. Prove that for a ray passing in a principal plane through a prism of refractive index μ from a medium of refractive index μ_1 into a medium of refractive index μ_2, the minimum deviation is given by

$$\mu_1\mu_2 \cos(D+i) = \mu^2 \cos i - \sqrt{(\mu^2-\mu_1^2)(\mu^2-\mu_2^2)},$$

when μ is greater than μ_1 and μ_2; and by

$$\mu_1\mu_2 \cos(i-D) = \mu^2 \cos i + \sqrt{(\mu_1^2-\mu^2)(\mu_2^2-\mu^2)},$$

when μ is less than μ_1 and μ_2.

14. Sunlight falls on a small isosceles prism standing on a horizontal table and emerges after reflection at the base, the edge of the prism being inclined at any angle to the sun's rays. Shew that the direction of the emergent rays is the same as if the light had been reflected at the table.

15. Parallel rays fall in any direction on the base of a right-angled prism, and after being reflected at the two faces containing the right angle, emerge through the base. Shew that their direction is opposite to that which they would have taken if they had been simply reflected at the principal plane of the prism.

16. Rays emanating from a point are incident on a prism whose refracting angle is a right angle. Prove that only those rays which fall on the part of the face of the prism contained between the refracting edge and a certain hyperbola will emerge from the other face; and shew that the eccentricity of this hyperbola is $1/\sqrt{\mu^2-1}$, where $\mu<\sqrt{2}$.

If $\mu>\sqrt{2}$ then the rays can only emerge through the base of the prism.

17. Rays are incident at a given point on a right-angled prism and form a cone of revolution about the normal at the point of incidence. Shew that the rays on emerging from the prism are parallel to the generators of a cone of revolution, and compare the angles of the two cones.

18. A ray is incident on a prism of angle i at a given angle in any plane; shew that, if it be reflected at the second face, the reflected ray makes a constant angle with a line in the principal plane inclined to the normals to the two faces at angles $2i$ and i respectively.

19. Rays are incident on a prism in a given plane but at any angle; shew that the emergent rays are parallel to the generators of a quadric cone, the circular sections of which are parallel to the plane of incidence and to a plane equally inclined with the plane of incidence to the principal plane.

20. Prove that with the usual notation the deviation of any ray passing through a prism is given by

$$\cos D = \mu\,\{\cos(\phi-\phi')+\cos(\psi-\psi')\}-\mu^2 \\ +(\mu\cos\phi'-\cos\phi)(\mu\cos\psi'-\cos\psi)\cos i.$$

A ray is incident at a given angle on one face of a prism of angle $\frac{1}{2}\pi$; prove that the deviation is least when the ray passes in a principal plane.

21. A ray just passes through a prism of angle i, being incident and emergent along the faces. Shew that its path in the prism makes with the edge an angle $\sin^{-1}\{\cos a\sec\frac{1}{2}i\}$, and that the deviation is

$$2\sin^{-1}\{\cot a\sin\tfrac{1}{2}i\},$$

where a is the critical angle.

22. Two equal prisms of angle i of the same substance are placed with a face of each in contact; shew that no ray can pass if the inclination of their edges exceed $2\sin^{-1}\left(\frac{\sin a}{\sin i}\right)$, where a is the critical angle.

23. A cube of refractive index μ is hung up in direct sunlight; shew that if the cosines of the angles the rays make with the edges of the cube all lie between $(2-\mu^2)^{\frac{1}{2}}$ and $(\mu^2-1)^{\frac{1}{2}}$, supposed real, the light will emerge from the cube in 56 directions. In how many directions can it emerge if the above conditions are not fulfilled?

If the cube be of glass, light can emerge in 8 directions.

24. A ray is refracted in any manner through a number of homogeneous media bounded by circular coaxial cylinders. Prove that the shortest distance between the axis and any portion of the ray is proportional to the cotangent of the angle the ray makes with the axis.

25. An origin of light is at a vertical height h directly over the edge of a circular pond of radius a; if the origin be taken at the point beneath the light, and the axis of x pass through the centre of the pond, and the axis of z be vertical, shew that the light in the water is bounded by the surface

$$z^2=(x^2+y^2-2ax)^2\{\mu^2h^2/4a^2x^2+(\mu^2-1)/(x^2+y^2)\}.$$

CHAPTER III.

GEOMETRICAL FOCI.

32. THROUGHOUT this chapter and elsewhere, when comparing lengths on the same straight line, we shall use the symbol AB, denoting the distance from a point A to a point B, as an algebraical expression. We therefore have $AB = -BA$, and if C be any point whatever on the same line as A and B,

$$AB = AC + CB = AC - BC = CB - CA.$$

As the formulæ are homogeneous in distances, measured algebraically along the same straight line, it is immaterial which direction be considered positive and which negative.

We confine our attention to pencils of rays which are all incident at small angles on reflecting or refracting spherical surfaces. The incidence is then said to be *direct*.

The angle subtended at its centre by the part of the surface on which the light is incident is the angular aperture, and is also a small quantity. The angles made by the rays with any radius to this part of the surface are therefore also small. The paths of the reflected or refracted rays will then be determined to a first approximation by the assumption that the squares of the circular measures of these angles may be neglected.

The accuracy of the result will depend on the numerical value of the second approximation, into which will enter the position and size of the object, the size of the surface, and also the position of the observer's eye in certain cases. This correction is evaluated generally in the chapters on Aberration and Distortion.

As a rule the error will not be appreciable if the angles enumerated above do not exceed 2° or 3°.

When the squares of the angular aperture and of the angles of divergence are neglected, certain points near the spherical surface must be treated as coincident, and certain lines as equal.

If Q be an origin of light, O the centre of a spherical surface, and if QO cut the surface in A, its vertex, the angle AOR is the angular aperture θ; the angle AQR is the angle of divergence of the ray QR from the axis OA, and is denoted by α.

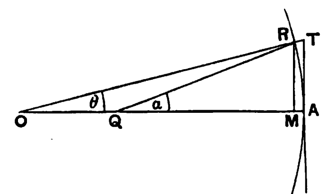


Fig. 18.

The ordinate $RM\,(=y)$ is the linear aperture; and we have, if r be the radius,

(i) $$y = r\sin\theta = r\theta$$

to the order of approximation adopted.

(ii) $$AM = r(1-\cos\theta) = \tfrac{1}{2}r\theta^2,$$

and therefore AM must be neglected.

Also, if QR cut the tangent plane at T, then $RT = AM\sec\alpha$ and must be neglected.

The spherical surface is therefore, as far as regards the positions of points of incidence, indistinguishable from the tangent plane at A, but the normals to the surface and to the plane make an angle θ with each other, which is of the first order and must be retained.

(iii) $$RQ = MQ\sec\alpha = (AQ - AM)\sec\alpha.$$

Hence, provided that the distance of Q from A be finite, *i.e.* that the angle α fall within our limits of approximation, AQ must be substituted for RQ.

33. Reflection at a Spherical Mirror.

Let O be the centre of a spherical mirror, Q an origin of light, and let QO cut the surface in the vertex A.

A ray QR will be reflected at R so that the angles of incidence

and reflection are equal. Since the plane of incidence contains

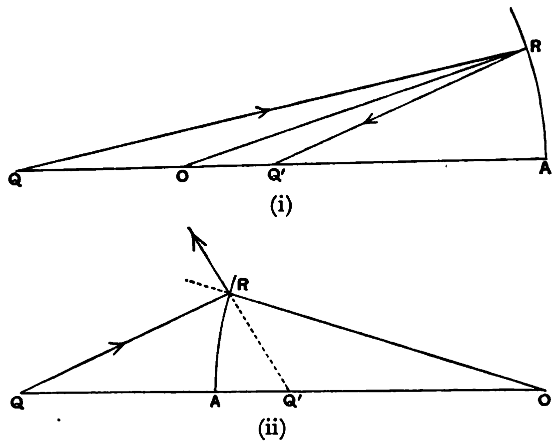


Fig. 19.

QA, whatever point on the spherical surface R may be, the reflected ray must intersect QA; and if it cut QA in Q', we have, for both concave and convex mirrors,

$$RQ' : RQ = Q'O : OQ.$$

Hence, to the order of approximation adopted,

$$AQ' . OQ = AQ . Q'O \quad \text{.....................} \quad \text{(i)},$$

i.e. $$AQ'(AQ - AO) = AQ(AO - AQ'),$$

or $$AQ . AO + AQ' . AO = 2AQ . AQ',$$

i.e. $$1/AQ' + 1/AQ = 2/AO \quad \text{........................} \quad \text{(I)}.$$

Rewriting the equation (i)

$$(OQ' - OA)\, OQ = (OQ - OA)(-OQ'),$$

i.e. $$OA . OQ + OA . OQ' = 2OQ . OQ',$$

or $$1/OQ' + 1/OQ = 2/OA \quad \text{.....................} \quad \text{(II)}.$$

These formulæ may easily be seen to be algebraic and to cover all cases; either by retracing the path of the light from Q' to Q, or by considering the light as if incident on the other side of the surface when on its way towards Q.

34. Geometrical Focus.

Since either formula (I) or (II) gives the same position for Q' corresponding to a given position of Q, whatever point R on the spherical surface be taken within the limits of approximation, it follows that all rays diverging from Q (up to a certain angle of divergence, dependent on the position of Q), pass after reflection through the same point Q'.

This point is called the *Geometrical Focus* of Q; also Q and Q' are said to be *conjugate foci.*

35. Formation of Images.

We have seen that, to the first order of approximation, every ray of a pencil that diverges from a point Q will after reflection converge to the geometrical focus Q' lying on QO. So also every ray of a pencil from P will converge to a geometrical focus P' lying on PO; and if $OP = OQ$, OP' will be equal to OQ'.

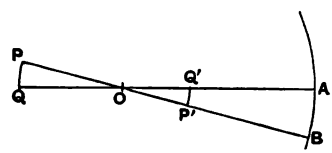


Fig. 20.

But if the rays of these two pencils are to be reflected at the same parts of the mirror, the angle AOB must fall within those limits whose square is to be neglected; and therefore the small circular arcs PQ, $P'Q'$ are indistinguishable from straight lines perpendicular to QO. The image of the small straight line PQ is therefore the small straight line $P'Q'$. This is true in whatever plane through QO the point P may lie, and we deduce the following proposition: *Every ray proceeding from any point in a small plane area perpendicular to the axis will, after reflection, pass through a point in a definite plane area, also perpendicular to the axis.*

In this proposition the term *axis* is used to denote the radius QO. In the case of a single mirror this may be understood as any radius, within the limits of this first approximation, to which the object is perpendicular; but when light is reflected directly between two spherical mirrors the term *axis* is restricted to the line joining their centres, and the proposition may be at once extended to the image formed after any number of reflections.

36. Principal Focus.

A small pencil of rays incident parallel to the axis OA will after reflection pass through the middle point of OA. This is obvious, since in this case we have $OQ' = Q'R = Q'A$, to our order of approximation.

This point is called the *principal focus* of the mirror, and is denoted by F. Also a pencil of rays, that diverge from F, will after reflection become a pencil of rays parallel to OA.

A plane through F perpendicular to the axis is called the *principal focal plane*, and any pencil of parallel rays, incident in a direction making a small angle with the axis, will after reflection pass through some point on this plane.

The distance AF is called the *focal length*, and is denoted by the symbol f.

In the equation $AF = FO = f$ a direction of positive measurement is tacitly assumed, and it is usual to reckon AF, and therefore f, as positive when measured from the point A in the same direction as the incident light travels, and negative when measured in the opposite direction.

The focal length is therefore positive for a mirror convex to the incident light, and negative for a mirror concave to the incident light. It is hardly necessary to point out that a surface convex to one side is concave to the other.

37. To construct the image of any small object PQ, which is perpendicular to the axis.

Determine the principal focus F; draw a ray PFR, this is reflected parallel to the axis at R.

Draw a ray PS parallel to the axis, this is reflected at S to pass through F.

These reflected rays meet in P', and $P'Q'$ is perpendicular to the axis. Also POP' is a straight line, since the ray PO, being incident normally, is reflected in the same straight line.

If in Fig. (i) Q be taken between F and A, then Q' will lie behind the mirror, and the figure in this case will be Fig. (ii) reversed, with PQ and $P'Q'$ exchanged.

It is obvious, since $PQ = SA$ and $P'Q' = RA$, that we have in all cases

$$\frac{P'Q'}{PQ} = \frac{FQ'}{FA} = \frac{FA}{FQ} \quad \text{.....................(III)},$$

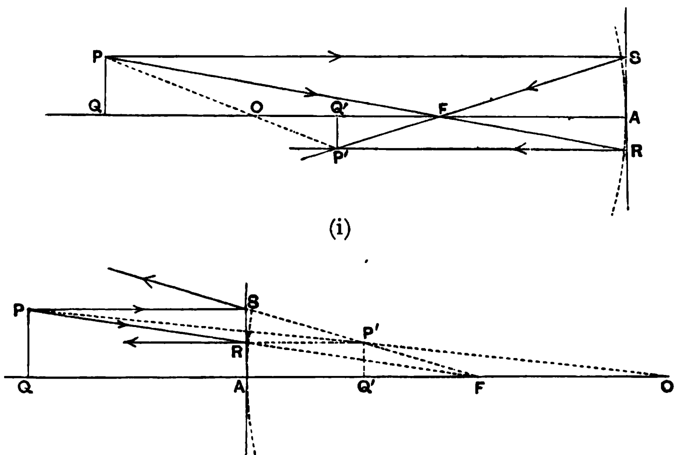


Fig. 21.

and that these equations are algebraic. When the image is inverted, that fact is expressed by the convention that $P'Q'$, PQ, being in opposite directions, have opposite signs, and therefore that FQ and FA, or FQ' and FA are in opposite directions and have opposite signs.

From (III) we deduce the formula

$$FQ \,.\, FQ' = FA^2 = f^2 \quad \text{................. (IV)}.$$

This is the simplest relation which connects the position of the conjugate foci, and it shews that Q and Q' always lie on the same side of F, and that if Q move along the axis, Q' will move along it in the opposite direction. Taking the case of a mirror concave to the incident light, in which the principal focus F lies in front of the mirror, we see that as Q moves from, say, positive infinity to F, Q' recedes from F to positive infinity. The reflected rays actually pass through Q', and $P'Q'$ is a *real* image. When Q moves from F to A, then Q' moves from negative infinity to A; the reflected rays do not actually pass through Q', and $P'Q'$ is said to be a *virtual* image.

If Q move on from A to negative infinity, PQ is a virtual object (in reality an image formed by previous reflections or refractions), and Q' moves from A to F, $P'Q'$ being a real image. The case of a convex mirror may be similarly treated.

In addition to (III) we may use the formulæ

$$\frac{P'Q'}{PQ} = \frac{OQ'}{OQ} = -\frac{AQ'}{AQ} \quad \text{.................. (V)};$$

but as each of these involves both object and image they are not so convenient. The distinction however, introduced by the minus sign in this formula according as O or A is the origin, will serve later to determine, in the case of a mirror equivalent to a whole series of reflections or refractions, which point is the centre and which the vertex.

38. Refraction at a Spherical Interface.

Let light be refracted at a spherical interface of centre O and vertex A from a medium of index μ into a medium of index μ'.

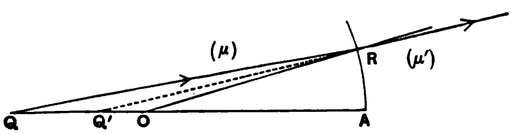


Fig. 22.

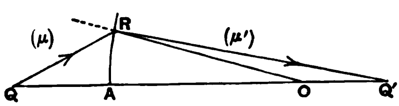


Fig. 23.

Then if QR be the incident ray, RQ' the refracted ray,

$$\mu \sin QRO = \mu' \sin Q'RO.$$

On dividing both sides of this equation by $\sin QOR$, we obtain

$$\mu OQ/RQ = \mu' OQ'/RQ'.$$

Hence to the order of approximation involved in neglecting the squares of the angles of incidence &c.,

$$\mu OQ \,.\, AQ' = \mu' OQ' \,.\, AQ \quad \text{....................(i)}.$$

Therefore

$$\mu(AQ - AO)\,AQ' = \mu'(AQ' - AO)\,AQ,$$

i.e.
$$\mu' AO \,.\, AQ - \mu AO \,.\, AQ' = (\mu' - \mu)\,AQ \,.\, AQ',$$

or
$$\frac{\mu'}{AQ'} - \frac{\mu}{AQ} = \frac{\mu' - \mu}{AO} \quad\text{.....................}\quad \text{(I)}.$$

Again, rewriting (i) we have

$$\mu OQ\,(OQ' - OA) = \mu' OQ'\,(OQ - OA).$$

Therefore

$$\mu' OA \,.\, OQ' - \mu OA \,.\, OQ = (\mu' - \mu)\,OQ \,.\, OQ',$$

or
$$\frac{1}{\mu' OQ'} - \frac{1}{\mu OQ} = \left(\frac{1}{\mu'} - \frac{1}{\mu}\right)\frac{1}{OA} \quad\text{..............}\quad \text{(II)}.$$

These formulæ are algebraic as before, and give the same point Q' conjugate to Q for all rays which do not pass the limits of approximation.

The point Q' is therefore the *Geometrical Focus* of Q; and as in Art. 35, the image of any small object PQ perpendicular to the axis is $P'Q'$, also perpendicular to the axis.

39. Principal Foci.

When pencils of rays are refracted directly at one or more spherical surfaces having a common axis, there are two cardinal points on that axis, called the principal foci, and defined as follows :—

The *first principal focus* F_1 is the point on the axis, which is the origin of a small pencil of rays that ultimately become parallel to the axis.

The plane through it perpendicular to the axis is the *first principal focal plane,* and rays from any point on this plane will ultimately become a pencil of parallel rays making a small angle with the axis.

The *second principal focus* F_2 is the point on the axis through which rays ultimately pass that are incident parallel to the axis; and the *second principal focal plane* is perpendicular to the axis at F_2, and contains the foci of all pencils of parallel rays incident at small angles with the axis.

In the case of refraction at one spherical surface, we obtain on putting Q' at infinity in (I) and (II),

$$AF_1 = -\frac{\mu}{\mu' - \mu} AO, \quad OF_1 = \frac{\mu'}{\mu' - \mu} OA.$$

And on putting Q at infinity, we obtain

$$AF_2 = \frac{\mu'}{\mu' - \mu} AO, \quad OF_2 = -\frac{\mu}{\mu' - \mu} OA.$$

40. To construct the image of any small object PQ, which is perpendicular to the axis.

Determine F_1 and F_2; draw the ray PF_1R, which will be refracted at R parallel to the axis. Draw the ray PS parallel to

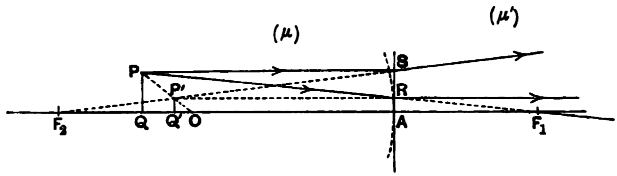


Fig. 24.

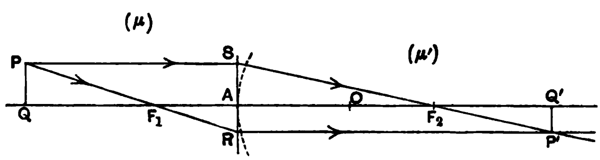


Fig. 25.

the axis; this will be refracted through F_2. These refracted rays intersect in P'; also P' lies on OP, since the ray PO, being incident normally, suffers no deviation. In all cases we have

$$\frac{P'Q'}{PQ} = \frac{F_2Q'}{F_2A} = \frac{F_1A}{F_1Q} \quad \text{.................. (III)},$$

and these equations are algebraic; while, as before, a negative value for the ratios implies that the image is inverted.

From (III) we deduce

$$F_1Q \,.\, F_2Q' = F_1A \,.\, F_2A = F_1O \,.\, F_2O$$

$$= -\frac{\mu\mu'}{(\mu' - \mu)^2} AO^2 \quad \text{.....................(IV)}.$$

Hence in all cases the distances F_1Q and F_2Q' lie in opposite directions, since they are of opposite signs, their product being negative.

The linear magnification is given by (III), and also by the equations

$$\frac{P'Q'}{PQ} = \frac{OQ'}{OQ} = \frac{AQ'/\mu'}{AQ/\mu} \quad\text{.....................}\quad (V).$$

The student should satisfy himself of the truth of the formulæ I—V, for all positions of the object and image, real or virtual, for surfaces convex or concave to the incident light, and for light refracted from a rarer into a denser medium, or conversely.

41. The Aplanatic points.

On any radius OA of a refracting spherical interface there are a pair of points I, I' such that all rays from I are refracted through I', whatever be the angle of incidence. These points are called the *aplanatic* points.

The distance OI is equal to $\frac{\mu'}{\mu} AO$, the distance OI' to $\frac{\mu}{\mu'} AO$.

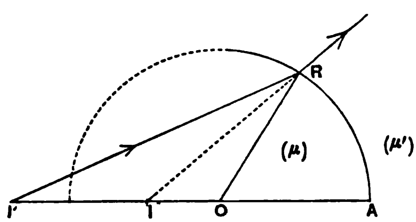


Fig. 26.

These points are therefore inverse points in the spherical surface, and if IR be any ray from I, the triangles OIR, ORI' are similar.

Hence

$$\sin ORI'/\sin ORI = \sin OIR/\sin ORI$$
$$= OR/OI = \mu/\mu'.$$

Therefore $I'R$ is the direction of the refracted ray, whatever be the angle of incidence at R.

If the refracting interface be convex to the incident light, then the points I, I' will fall behind the surface, and I will be a virtual and I' a real focus. For example in the figure a ray in the

second medium on its way towards I' will be refracted accurately to I.

These points I, I' may also be used to determine geometrically the image of any small object PQ which is perpendicular to the radius OA. For if IP cut the refracting surface at R (which will be indistinguishable from a point on the tangent plane at A), the refracted ray is $I'R$, and $OP, I'R$ must intersect in the geometrical focus P'.

The formula $\dfrac{P'Q'}{PQ} = \dfrac{\mu' I'Q'}{\mu IQ}$ is easily deduced from this construction.

42. Construction for conjugate points.

Since $F_1Q \,.\, F_2Q' = F_1A \,.\, F_2A$, it follows that points on the axis, considered as the origins of incident pencils, form a range, which is projective with regard to the range formed by their geometrical foci. The two self-conjugate points of the ranges are A and O; also F_1 and ∞ in the first range are conjugate respectively to ∞ and F_2 in the second range, and I in the first range is conjugate to I' in the second.

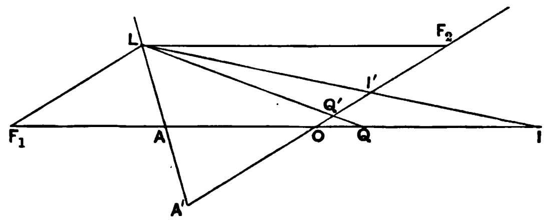


Fig. 27.

Take then two equal lines OA, OA' inclined to each other at any angle, and determine the points I and I' one on each line at distances $\frac{\mu'}{\mu} AO$ and $\frac{\mu}{\mu'} AO$ from O respectively. If AA', II' meet in L, the line LQ to any point on AO will meet $A'O$ in the point Q' at the same distance from O as the geometrical focus of Q.

Also a line through L parallel to $A'O$ meets AO in the first principal focus F_1, and a line parallel to AO meets $A'O$ at the distance of the second principal focus F_2.

43. Refraction at a Plane Interface.

This may be regarded as a special case of the spherical surface, the radius being infinite.

Or directly, since

$$\mu \sin RQA = \mu' \sin RQ'A,$$
$$\mu RA/RQ = \mu' RA/RQ',$$

i.e. within our limits of approximation

$$AQ'/\mu' = AQ/\mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}.$$

Since the *geometrical focus* Q' lies on the same normal to the

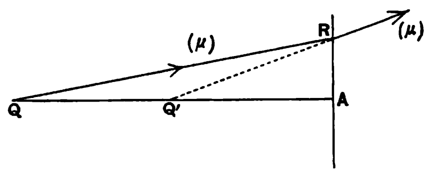


Fig. 28.

plane as Q, the image of any small object parallel to the plane is equal to it and also parallel to the plane.

44. Example.

To find the geometrical focus of a small pencil after direct refraction through a sphere of refractive index μ and radius a, and to construct the image geometrically.

Let an origin of light Q lie on the diameter AB, and let Q_1 be the

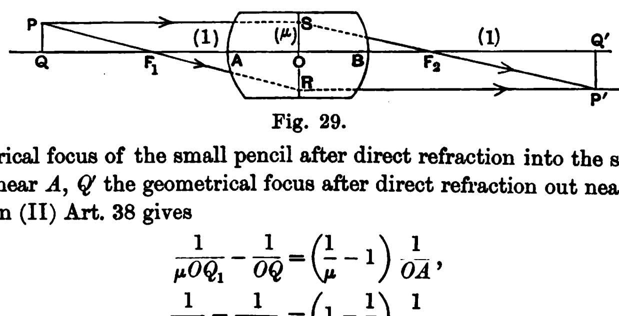


Fig. 29.

geometrical focus of the small pencil after direct refraction into the sphere at points near A, Q' the geometrical focus after direct refraction out near B.

Then (II) Art. 38 gives

$$\frac{1}{\mu OQ_1} - \frac{1}{OQ} = \left(\frac{1}{\mu} - 1\right)\frac{1}{OA},$$
$$\frac{1}{OQ'} - \frac{1}{\mu OQ_1} = \left(1 - \frac{1}{\mu}\right)\frac{1}{OB}.$$

Since $OB = -OA$, we obtain by adding these equations,

$$\frac{1}{OQ'} - \frac{1}{OQ} = \frac{2(1-\mu)}{\mu OA}.$$

To determine the principal foci we put $OQ' = \infty$ in this equation, and obtain $OF_1 = \frac{\mu}{2(\mu-1)}OA$; if we put $OQ = \infty$, we obtain $OF_2 = -\frac{\mu}{2(\mu-1)}OA$. These foci are therefore at equal distances from O, and lie outside the sphere if $\mu < 2$.

Again, let l, l' denote the linear dimensions of a small object PQ perpendicular to AB and of its image $P'Q'$. Then

$$l'/l = \frac{OQ'}{OQ_1} \cdot \frac{OQ_1}{OQ} = OQ'/OQ.$$

Hence, if $l'=l$, we deduce from this equation, together with the equation above, that Q and Q' coincide at O. In other words, a ray, which is on its way to any point R on the plane through O perpendicular to AB, will after refraction through the sphere continue as if from R.

To construct then the image of any object PQ, draw the ray PF_1R which emerges parallel to AB as if from R, and the ray PS parallel to AB, which emerges through F_2. These rays meet in P', and it is obvious from the similar triangles of the figure that

$$\frac{P'Q'}{PQ}=\frac{F_1O}{F_1Q}=\frac{F_2Q'}{F_2O};$$

and therefore that

$$F_1Q\,.\,F_2Q'=F_1O\,.\,F_2O=-\left\{\frac{\mu a}{2(\mu-1)}\right\}^2.$$

45. Example.

A plane mirror is placed behind a sphere of radius a and refractive index μ. Shew that the effect on a small pencil, which passes directly through the sphere, is reflected at the plane, and re-traverses the sphere, will be exactly the same as if it had been reflected at a certain mirror.

Let M be the given plane mirror, F_1, F_2 the principal foci of the sphere as determined in the previous example.

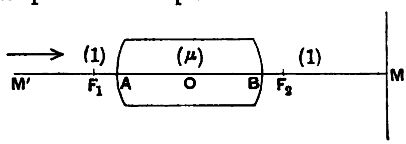


Fig. 30.

Then if Q be the origin of light on AB, Q_1 its geometrical focus after refraction through the sphere, Q_2 the image of Q_1 in the plane, and Q' the geometrical focus after traversing the sphere again, we have

$$F_1Q\,.\,F_2Q_1=F_1O\,.\,F_2O,$$
$$F_2Q_1+F_2Q_2=2F_2M,$$
$$F_2Q_2\,.\,F_1Q'=F_1O\,.\,F_2O.$$

Hence
$$\frac{1}{F_1Q}+\frac{1}{F_1Q'}=\frac{2F_2M}{F_1O\,.\,F_2O}=\frac{2}{F_1M'},$$

where M' is the image of M formed by the sphere.

Also the linear magnification l'/l is the product of the three linear magnifications F_1O/F_1Q, 1, and F_1Q'/F_1O, in the sphere, the plane, and the sphere respectively; hence $l'/l=F_1Q'/F_1Q$.

It therefore follows by comparing these equations with (II) Art. 33, and (V) Art. 37 that F_1 is the centre and M' the vertex of the hypothetical mirror which produces the same effect on a small direct pencil as the given combination.

These results may be at once obtained graphically by using the geometrical constructions of Art. 40 for the paths of rays from any point P.

EXAMPLES.

1. A small pencil passes directly through a series of parallel plates of thicknesses $t_1, t_2 \dots t_n$, and refractive indices $\mu_1, \mu_2, \dots \mu_n$, the refractive index of the medium on either side of the system being μ_0. Shew that the geometrical focus is at a distance $\sum_{r=1}^{r=n}\left(\frac{\mu_r - \mu_0}{\mu_r} t_r\right)$ from the origin of the pencil.

2. Two concave reflectors of radii a and b are placed facing each other, their vertices being at the distance $\frac{1}{2}(a+b)$ apart. Prove that the linear magnification of any small object between them perpendicular to their common axis, which is produced by $2n$ reflections, is $(-b/a)^n$ or $(-a/b)^n$, according as the first reflection takes place at the mirror of radius a or at that of radius b.

3. Two concave reflectors of radii a and b are placed facing each other, their centres coinciding. Shew that the image formed by $(2n+1)$ reflections is the same as if the rays were directly reflected at a concentric spherical mirror of curvature $(n+1)/a+n/b$.

Give a geometrical construction for the image formed after an even number of reflections.

4. A plane mirror is placed facing a spherical mirror and perpendicular to its axis. The image of any object formed by two reflections in the plane mirror and one in the spherical mirror is the same as the image formed by a single reflection in a spherical mirror, having its focus and vertex at the images in the plane of the focus and vertex of the spherical mirror. The image formed by two reflections in the spherical and one in the plane mirror is the same as that formed by a single reflection in a spherical mirror with its centre at the focus, and its vertex at the image of the plane in the spherical mirror.

5. A plane mirror is placed at M perpendicular to the axis of a spherical mirror of focus F and vertex A. Shew that the image of a small object, which is formed by $(n+1)$ reflections at the plane and n reflections at the spherical mirror is the same as that formed by a single reflection at a spherical mirror of focal length $FA \sin \alpha \operatorname{cosec} n\alpha$, having its focus at distance $FA \sin (n+1)\alpha \operatorname{cosec} n\alpha$ from F; and that the image formed by $(n+1)$ reflections at the spherical mirror and n reflections at the plane is the same as that formed by a single reflection at a spherical mirror of focal length $AF \sin \alpha \operatorname{cosec} (n+1)\alpha$, with its focus at distance $FA \sin n\alpha \operatorname{cosec} (n+1)\alpha$ from F, where $\cos \alpha = FM/FA$, and the distances are algebraic.

6. Shew that, if two spherical mirrors be placed on the same axis, there are two points on the axis which are conjugate foci in both mirrors; and that these points are real only when the distance between the foci of the mirrors does not lie between the sum and difference of their numerical focal lengths.

Shew further that when these points are real the geometrical focus of any point whatever on the axis tends to coincide with one or other of them, as the number of reflections between the mirrors continually increases.

7. Determine the point B on the axis of a refracting spherical interface for which the linear magnification is -1, and shew that, if B' be its conjugate, and Q, Q' any pair of conjugate foci,

$$\mu/BQ - \mu'/B'Q' = (\mu' - \mu)/AO.$$

8. A small pencil passes directly through a sphere of radius r and refractive index μ_1 from a medium of index μ_0 into one of index μ_2; shew that the effect is the same as if it were refracted from μ_0 into μ_2 at a single concentric spherical surface of radius

$$r\mu_1(\mu_2 - \mu_0)/\{\mu_1(\mu_0 + \mu_2) - 2\mu_0\mu_2\}.$$

An object in water is seen through a glass sphere meeting the water; draw the course of the pencil by which it is viewed, distinguishing the cases when the distance of the object from the centre of the sphere is greater or less than $9r/5$.

9. If any number of refracting media are separated by complete concentric spheres, shew that for light passing completely through, the principal foci F_1, F_2 of this refracting system are at equal distances from the centre, and that the linear magnification $= F_2Q'/F_2O = F_1O/F_1Q$, where O is the centre of the sphere.

10. Any number of media are bounded by concentric spherical shells, and the back of the outermost shell is silvered; shew that for any pencil incident directly the equation $1/p + 1/q = \text{constant}$ connects the distances from the centre of any origin of light and its geometrical focus; and that the combination is exactly equivalent to a concentric spherical mirror.

11. A small pencil passes directly through a hemisphere of refractive index μ placed in a medium of index μ_0, having its plane face towards the incident light. Shew that, if O be the centre and A the vertex of the hemisphere, the equation

$$\mu^2/OQ' - \mu_0^2/OQ = (\mu - \mu_0)\mu/OA$$

connects the distances from the centre of any origin of light Q and its geometrical focus Q'.

Determine the points of unit magnification, and the principal foci.

12. A small pencil is refracted directly through a sphere, centre O, of which the first hemisphere, vertex A, is of refractive index μ_1 and the second hemisphere is of refractive index μ_2. The sphere is placed in a medium of index μ_0. Obtain the equation

$$\mu_2^2/OQ' - \mu_1^2/OQ = \{\mu_0(\mu_1+\mu_2) - \mu_1^2 - \mu_2^2\}/OA.$$

Determine the points of unit magnification H and H'; and shew that

$$1/H'Q' - 1/HQ = \{(\mu_0 - \mu_1)/\mu_2 + (\mu_0 - \mu_2)/\mu_1\}/OA.$$

13. Two spherical surfaces on the same axis are such that the vertex of the first is the centre of the second; they form the boundary of a medium of index μ. If A, B be the two vertices, and O the centre of the first surface, obtain the equation

$$\mu^2/AQ' - 1/AQ = (\mu - 1)(\mu/AB + 1/AO),$$

connecting the focus and origin of any small pencil passing through directly.

Determine the principal foci F_1, F_2, and shew that

$$F_1Q \cdot F_2Q' = -\{(\mu - 1)(1/AB + 1/\mu AO)\}^{-2}.$$

14. The front of a piece of glass is plane, the back is spherical and silvered. Shew that for small pencils incident directly the effect is the same as direct reflection at a spherical surface of radius two-thirds that of the spherical boundary, with its vertex at one-third of the thickness in front of the vertex.

15. The surfaces of a sheet of glass are portions of concentric spheres of centre O, and the convex surface of the sheet is silvered. A pencil from any point on the axis of the sheet is incident directly on the concave surface. The points Q, Q', Q'' are the geometrical foci of those portions of the light which have suffered respectively one reflection and no refraction, one reflection and two refractions, three reflections and two refractions. Prove that the range $OQQ'Q''$ is harmonic.

16. A luminous point is placed on the line of centres between two refracting spheres of the same substance ($\mu = \frac{4}{3}$) and equal radii r, which are silvered on the sides remote from the luminous point. A pencil of light passes in and out of each sphere in succession. Shew that, if u, v are the distances of the luminous point and the final geometrical focus from the silvered surface of the first sphere, c the distance between the silvered surfaces,

$$v = c - \frac{r^2}{c-}\,\frac{r^2}{u}.$$

17. Prove that the image of a small straight line, formed by geometrical foci after refraction at a spherical surface, is a small straight line; and shew that, if the line and its image make angles ϵ, ϵ' with the axis,

$$\tan\epsilon'/\tan\epsilon = 1/m.$$

Prove that the magnification of the line is $m(\sin^2\epsilon + m^2\cos^2\epsilon)^{\frac{1}{2}}$, where m is the linear magnification for the point where it cuts the axis.

18. An origin of light is situated within a solid sphere of glass of radius a at distance a/μ from the centre. Prove that all rays which emanate from it will after refraction appear to come from a certain point P, or else proceed as if they had started from P and been reflected at the convex surface of the sphere.

19. An origin of light is placed outside and in contact with the plane base of a glass hemisphere of radius a at distance a/μ from the centre.

Shew that the light, which emerges from the hemisphere at points lying on the same side as the centre of the plane through the point perpendicular to the radius, will meet a screen placed parallel to the base within an area bounded by arcs of a circle and a hyperbola.

20. The point O is the centre of a spherical refracting surface, separating media of indices 1 and μ, and QR is a ray incident at R. The points L, M divide OR internally and externally in the ratio $\mu : 1$; prove that if QR, or QR produced, as the ray is incident on the convex or concave side of the sphere, cut the circle whose diameter is LM in N, then ON is parallel to the refracted ray.

21. Prove the following construction for the geometrical focus Q' of a small pencil, having Q as origin, and directly refracted at the point A of a spherical surface, whose centre is O, separating media of indices 1 and μ. Join an arbitrary point C to O and A; draw a line parallel to CQ to cut CO, CA in O' and A' respectively; produce $A'O'$ to q' so that

$$O'q' = A'O'/(\mu - 1),$$

then Q' is on Cq'.

CHAPTER IV.

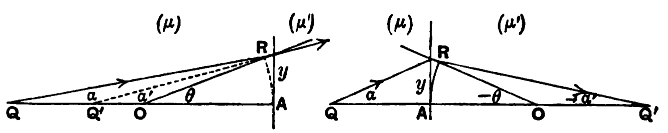
LENSES.

46. WHEN a small pencil of light is refracted directly at several spherical surfaces in succession, the final position of the geometrical focus may be determined by the formulæ of the preceding chapter; but since either the vertices or centres, or, better still, the principal foci of successive surfaces must be taken as origins, the resulting formulæ are very complicated. It is simpler to make use of transformations of (I) Art. 38, which contain the successive angles of divergence of the pencil of rays, and which may be found directly as follows:—

The straight line on which the centres of the successive spherical interfaces lie is called the axis; it is assumed that all the rays throughout make only small angles with this line. Let a small pencil of rays diverge from a point on this axis; and let α be the angle between the axis and a ray of this pencil. *This angle of divergence α is to be reckoned positive if the pencil of rays be divergent, and negative if the pencil be convergent.*

The curvature of the spherical interface is to be reckoned positive if the surface be concave towards the side on which the light is incident.

With these conventions let a ray QR making an angle α with



(i) (ii)

Fig. 31.

the axis QA be refracted at a spherical interface of curvature $1/\rho$ from a medium of refractive index μ into a medium of index μ'.

Let $Q'R$ be the refracted ray, and let $AQ'R = \alpha'$; then α' is the angle of divergence of the refracted ray. Let the angular aperture AOR be θ, and let the radius of the aperture AR be y. Then the angles of incidence and refraction are $\theta - \alpha$, $\theta - \alpha'$ respectively. We have therefore

$$\mu \sin(\theta - \alpha) = \mu' \sin(\theta - \alpha').$$

If then, as before, we neglect in our equations terms which are to those retained as the squares of the circular measures of the angles of incidence &c. are to unity, this equation gives

$$\mu(\theta - \alpha) = \mu'(\theta - \alpha'),$$

i.e.
$$\mu'\alpha' - \mu\alpha = (\mu' - \mu)\theta = \frac{\mu' - \mu}{\rho} y \quad \text{.............. (I).}$$

With the same approximations as before we have

$$y = \alpha \,.\, AQ = \alpha' \,.\, AQ' = \theta \,.\, AO,$$

and we deduce the equation (I), Art. 38,

$$\mu'/AQ' - \mu/AQ = (\mu' - \mu)/AO.$$

Hence it follows as in the preceding chapter that all the refracted rays for which α' does not exceed a certain limit, will diverge from the same point Q' on the axis.

The coefficient $(\mu' - \mu)/\rho$ of y in (I) is called the *power* of the surface, and is denoted by κ.

It will be found that, with the above conventions as to the signs of α, α', and ρ, the formula connecting α' and α holds in all cases.

47. Helmholtz's formula for linear magnification*.

It has been shewn (Art. 38) that the image of any small object at Q, which is perpendicular to the axis, will be found at Q', and will also be perpendicular to the axis. Also, if l, l' be the linear dimensions of the object and the image respectively, $l/l' = OQ/OQ'$, since any two conjugate foci lie on the same radius. But if R be the point of incidence of a ray for which the angles of divergence before and after refraction are α and α',

$$OQ/OR = \sin\phi/\sin\alpha,$$

$$OQ'/OR = \sin\phi'/\sin\alpha'.$$

Hence
$$\mu l \sin\alpha = \mu' l' \sin\alpha'.$$

* Helmholtz, *Wissenschaftliche Abhandlungen*, Bd. II. p. 189.

This equation is accurately true for finite values of α and α', when the rays from the origin are brought exactly to a focus, as in the case of the aplanatic points of Art. 41, but in all other cases we must neglect the squares of α and α', and the equation is

$$\mu l \alpha = \mu' l' \alpha' \quad \text{.........................(II)}.$$

If the image be inverted l and l' must have opposite signs, but an inspection of the figure will shew that the image has then been formed by convergent pencils, for which α' has been defined as a negative quantity.

Corollary. For any number of direct refractions at spherical surfaces with their centres lying on a common axis, the product $\mu l \alpha$ is constant, where α denotes the angle of divergence of that part of a ray which lies in the medium of refractive index μ, and l is the linear dimensions of the small image formed by the rays in their passage through that medium.

48. Linear Magnification at a single refraction.

With the above notation we have

$$y = \alpha \,.\, QA,$$

$$\mu'\alpha' - \mu\alpha = (\mu' - \mu)\, y/\rho = \kappa y.$$

Hence $$\mu'\alpha'/\mu\alpha = l/l' = 1 + \kappa QA/\mu \quad \text{.....................(i)}.$$

This formula will hold for all positions of Q, provided we reckon QA *positive when measured onwards with the light from* Q; but if, for example, the pencil be converging towards Q when incident on the surface, then both α and QA will be negative.

The position of the first principal focus is given by putting α' zero in (i); *i.e.* we have

$$0 = 1 + \kappa F_1A/\mu \quad \text{.......................(ii)},$$

and therefore by substitution from (ii) in (i),

$$l/l' = QF_1/AF_1.$$

Again, if we take Q' as the origin of light and retrace the path of a ray, we may rewrite (i) as

$$\mu\alpha/\mu'\alpha' = l'/l = 1 + \kappa\, Q'A/\mu' \quad \text{.............. (iii)}.$$

In this formula the sign of κ is unaltered, for the signs of both

numerator and denominator are changed, since the light in retracing its path is now incident on the convex side when previously it was incident on the concave side, and conversely.

Also the distance $Q'A$ *must be reckoned positive if measured from Q' with the light retracing its path.* This direction of measurement is necessarily in the opposite direction to that of QA.

Putting α zero in (iii) the position of F_2 is given by

$$0 = 1 + \kappa F_2A/\mu' \qquad\text{(iv)},$$

and the linear magnification is therefore given by

$$l'/l = Q'F_2/AF_2.$$

Hence $$F_1Q \,.\, F_2Q' = F_1A \,.\, F_2A$$

$$= -\mu\mu'/\kappa^2 \qquad\text{(v)},$$

since F_1A, F_2A as given by (ii) and (iv) are necessarily in opposite directions.

LENSES.

49. A lens is a portion of some refractive medium bounded by two spherical surfaces. The line joining their centres is the axis of the lens; and we suppose that all rays passing through the lens make only very small angles with the axis.

Let the axis cut the lens in the two vertices A and B; we must consider any ray as meeting each spherical surface and the tangent plane at its vertex in coincident points.

Let a ray from a point Q on the axis traverse the lens, and cross the surfaces at distances y_1, y_2 from the axis, and let its successive angles of divergence be α, α_1, and α'.

If, as is usually the case, the lens be placed in air, and if μ be the refractive index of its material, ρ and σ the radii of curvature of the first and second surfaces respectively, reckoned positive when these surfaces are concave to the light in its passage through the lens, the powers κ_1, κ_2 of the surfaces will be given by the equations

$$\kappa_1 = (\mu - 1)/\rho \quad\text{and}\quad \kappa_2 = (1 - \mu)/\sigma.$$

Let t be the thickness AB of the lens, which is generally small compared with ρ and σ.

We have, to the order of approximation adopted throughout, the equations

$$y_1 = \alpha . QA \quad \text{(i)}$$

$$\mu\alpha_1 - \alpha = \kappa_1 y_1 \quad \text{(ii)}$$

$$y_2 - y_1 = \alpha_1 . AB = \mu\alpha_1 . t/\mu \quad \text{(iii)}$$

$$\alpha' - \mu\alpha_1 = \kappa_2 y_2 \quad \text{(iv)}.$$

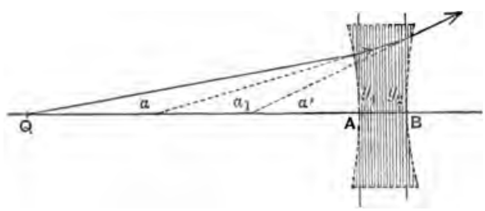


Fig. 32.

Substitute from (i) in (ii),

$$\mu\alpha_1/\alpha = 1 + \kappa_1 QA.$$

Substitute from (i) and this equation in (iii),

$$y_2/\alpha = QA + AB/\mu + \kappa_1 QA . AB/\mu,$$

and finally, substituting in (iv),

$$\alpha'/\alpha = 1 + \kappa_1 QA + \kappa_2 (QA + AB/\mu) + \kappa_1\kappa_2 QA . AB/\mu \quad \longrightarrow \quad \text{(I)}.$$

In this formula, as before, QA must be considered an algebraical quantity to be reckoned *positive if measured onwards with the light from Q, i.e.* if Q lie in front of the lens.

To obtain the corresponding formula for α/α', follow the reversed path of a ray from Q'; then

$$\alpha/\alpha' = 1 + \kappa_2 Q'B + \kappa_1 (Q'B + BA/\mu) + \kappa_2\kappa_1 Q'B . BA/\mu \quad \longleftarrow \quad \text{(II)}.$$

Here it is inevitable that $Q'B$ be reckoned *positive when taken with the light reversed, i.e.* if Q' lie behind the lens.

The ray from Q' will now be incident first near B, and secondly near A. The order of the terms in (I) is therefore altered, but for the reasons given in Art. 48, κ_1 and κ_2 will not be altered in sign.

We shall use the arrow drawn from left to right to indicate the direction of the incident light, and the standard direction in which the distance from the origin of light is measured; the arrow

from right to left will then indicate the direction of the light reversed, and the standard direction of measurement from the geometrical focus. It is only in formulæ of the type (I) and (II) that this convention need be made; in all other formulæ the convention that distances on the same straight line are treated as algebraic quantities will be found sufficient.

Since by Helmholtz's formula $l\alpha = \mu l_1 \alpha_1 = l'\alpha'$, formula (I) is an expression for l/l', that is, for the reciprocal of the linear magnification of any small object perpendicular to the axis; and formula (II) is an expression for l'/l, that is, for the linear magnification.

It must not be thought that it is only points on the axis from which the rays after refraction through the lens converge to a point. The considerations of Art. 35 shew that all rays from points within a certain small distance from the axis will after each refraction pass through a point; and therefore that the image of a point formed by a lens is a point. The whole path of any given ray however is not in one plane unless it cut the axis, and therefore needs the two equations given below, Art. 56, to specify it completely, instead of (I) or (II) only.

50. Points and Planes of Unit Magnification.

If in formulæ (I) and (II) we put $l = l'$, we can determine uniquely the positions of the two corresponding points H and H'. These points are called the *Unit Points**, and are such that any small object placed perpendicularly to the axis at the first unit point H has its image equal in magnitude and similarly placed at the second unit point H'.

Planes through H and H' perpendicular to the axis are called the *Unit Planes*; and the line joining the point in which any incident ray cuts the first unit plane to the point in which the emergent ray cuts the second unit plane is parallel to the axis.

Since the lens has the same medium on both sides, the equality $l = l'$ involves the equality $\alpha = \alpha'$; *i.e.* any ray crossing the axis at H will after traversing the lens cross the axis at H' with its direction unaltered.

* These points and planes have been called the principal points and planes; it seems better to avoid the confusion with the principal foci and focal planes by using the word *unit* to express their distinctive property.

This property ($\alpha = \alpha'$) is the definition of the *Nodal Points*; and we see that only in the case of the first and last media being the same, do the unit points and nodal points coincide.

51. Power of a Lens.

Formula (I) may be written in the form

$$\alpha'/\alpha = 1 + \kappa_2 t/\mu + (\kappa_1 + \kappa_2 + \kappa_1\kappa_2 t/\mu)\, QA. \longrightarrow$$

Hence by the definition of H

$$1 = 1 + \kappa_2 t/\mu + (\kappa_1 + \kappa_2 + \kappa_1\kappa_2 t/\mu)\, HA.$$

And subtracting, since

$$QA \equiv QH + HA,$$

$$\alpha'/\alpha - 1 = (\kappa_1 + \kappa_2 + \kappa_1\kappa_2 t/\mu)\, QH.$$

The expression $\kappa_1 + \kappa_2 + \kappa_1\kappa_2 t/\mu$ is called the *power* of the lens, and is denoted by K; in terms of the radii it is

$$\left\{(\mu - 1)\left(\frac{1}{\rho} - \frac{1}{\sigma}\right) - \frac{(\mu-1)^2 t}{\mu\rho\sigma}\right\}.$$

Let the distance from the axis of the point in which an incident ray cuts the first unit plane be y; then y will also be the distance from the axis of the point in which the emergent ray cuts the second unit plane; and as the distance QH is measured onwards with the light, we have $y = \alpha \,.\, QH$.

Hence the equation above gives

$$\alpha' - \alpha = Ky \quad\text{......................}\quad \text{(III)}.$$

The deviation of any ray which meets the axis is therefore proportional to the power of the lens; and lenses are divided into two classes according as the deviation is from or towards the axis.

If K be positive, the lens will increase the divergence of a pencil, and it is called a *Divergent Lens*; if K be negative, then the divergence will be decreased, or a convergent pencil will be made more convergent, and the lens is called a *Convergent Lens**. The terms *Dispersive* and *Collective* are also used.

* Divergent lenses have been called concave and also positive, convergent lenses convex and also negative.

The sign of K, or the nature of a lens, is at once seen if it be placed in direct sunlight. A convergent lens brings the rays to a focus behind it, and acts as a burning-glass; for a divergent lens the second principal focus is virtual and in front of the lens, and the sun's rays diverge after passing through the lens.

52. Principal Foci and Focal Length.

The principal foci have been defined in Art. 39. They lie on the axis; and F_1 is given by the condition that any ray from F_1 ultimately becomes parallel to the axis, *i.e.* for this point α' vanishes; while rays from any point on the first focal plane ultimately become a pencil of parallel rays.

The second principal focus F_2 is the point on the axis in which rays incident parallel to the axis ultimately meet, *i.e.* for this point α is zero; the second focal plane contains the foci of all pencils of parallel rays initially making any small angle with the axis.

In (I) put Q at H and F_1 successively; then

$$1 = 1 + \kappa_1 HA + \kappa_2 (HA + AB/\mu) + \kappa_1\kappa_2 HA \,.\, AB/\mu, \quad \longrightarrow$$

$$0 = 1 + \kappa_1 F_1A + \kappa_2 (F_1A + AB/\mu) + \kappa_1\kappa_2 F_1A \,.\, AB/\mu. \quad \longrightarrow$$

Hence on subtraction

$$1 = (\kappa_1 + \kappa_2 + \kappa_1\kappa_2 t/\mu)\, HF_1 \quad \longrightarrow \qquad \text{(IV)}.$$

We see then that HF_1 is the reciprocal of the quantity K defined above as the power of the lens.

The distance HF_1 is defined as the *Focal Length*, f, of the lens; f is positive if K be positive, and will then be measured from H towards F_1 in the direction in which the light traverses the lens.

In the same way that (I) has been used to determine the distances of H and F_1 from the first surface of the lens, we may use (II) to find the distances of H' and F_2 from the second surface.

On subtraction we have

$$1 = (\kappa_1 + \kappa_2 + \kappa_1\kappa_2 t/\mu)\, H'F_2, \quad \longleftarrow$$

but now $H'F_2$ is necessarily measured in the direction of the light retracing its path.

Hence in all cases HF_1 and $H'F_2$ are equal in magnitude and opposite in direction (cf. § 54).

53. Formula for Linear Magnification.

We may write (I) as

$$l/l' = 1 + \kappa_2 t/\mu + K\,.\,QA. \quad \longrightarrow$$

Also $$0 = 1 + \kappa_2 t/\mu + K\,.\,F_1A.$$

Hence $$l/l' = K\,.\,QF_1 = QF_1/HF_1 \;\ldots\ldots\ldots\ldots\ldots \text{(V)}.$$

Also from (II), $$l'/l = 1 + \kappa_1 t/\mu + K\,.\,Q'B \quad \longleftarrow$$

$$0 = 1 + \kappa_1 t/\mu + K\,.\,F_2B,$$

and therefore $$l'/l = K\,.\,Q'F_2 = Q'F_2/H'F_2 \ldots\ldots\ldots\ldots\ldots \text{(V)}.$$

Where equations (V) involve ratios of lengths, it is unnecessary to specify the direction of measurement provided the distances be treated as algebraic; and we deduce that in all cases

$$F_1Q\,.\,F_2Q' = F_1H\,.\,F_2H' = -f^2 \;\ldots\ldots\ldots\ldots \text{(VI)}.$$

The distances F_1Q, F_2Q' are therefore measured in opposite directions; and if Q approach F_1, Q' will recede from F_2, *i.e.* Q and Q' move along the axis in the same direction.

On substituting in (VI),

$$F_1Q \equiv F_1H + HQ \text{ and } F_2Q' \equiv F_2H' + H'Q',$$

we can deduce the formula

$$\frac{1}{H'Q'} - \frac{1}{HQ} = \frac{1}{H'F_2} = \frac{1}{F_1H} \;\ldots\ldots\ldots\ldots \text{(VII)},$$

which is also contained in (III).

We have assumed in formulæ III—VII that K is not zero, nor f infinite, as the lens would then be optically worthless, and no such points as the foci exist (cf. Art. 72).

54. Geometrical Constructions.

The formulæ of Arts. 51 and 53 may with advantage be found geometrically, provided that the unique existence of the principal foci and of the unit points be assumed.

Taking any arbitrary positions on the axis for F_1, H and H', draw two rays which are parallel before incidence and whose directions, real or virtual, cross the axis at H and F_1 respectively. Then on emergence the first ray, either really or virtually, passes through H', and is parallel to its original direction; the second

ray, which before incidence meets the first unit plane, either really or virtually, in R, will be parallel to the axis, and meet really or virtually the second unit plane in R', where $H'R' = HR$.

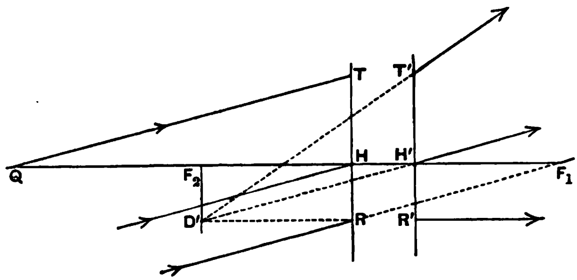


Fig. 33.

Since the incident rays are parallel, the focus, real or virtual, of the emergent rays is on the second principal focal plane; hence, if they intersect in D', $D'F_2$ is perpendicular to the axis, and it is obvious from the equal triangles $F_2H'D'$ and HF_1R that

$$F_2H' = HF_1.$$

Again, let QT be any incident ray parallel to these two rays, which crosses the axis at Q and meets the first unit plane in T. The emergent ray is $D'T'$, where $H'T' = HT$. Since $D'H'$ is parallel to the incident ray QT, the deviation produced by the lens is obviously the angle $H'D'T'$, which is, to our order of approximation, equal to $H'T'/F_2H'$, *i.e.* to Ky, with the notation of Art. 51.

The above figure is drawn for a divergent lens; for a convergent lens the figure would be as in Fig. 34. In either case, to

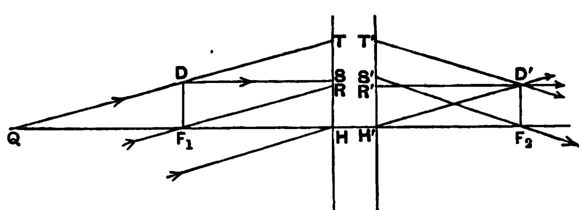


Fig. 34.

draw the emergent ray answering to the incident ray QT, we might make use of the point D in which QT cuts the first focal

plane. A ray DS parallel to the axis emerges in the direction $S'F_2$, and the emergent ray answering to QT is parallel to $S'F_2$.

As above the deviation produced by this lens is the angle $H'D'T'$, but is considered negative, being towards the axis, and its algebraical value is still Ky, for in the second figure

$$F_1H = H'F_2 = -1/K.$$

To find the image of any small object PQ.

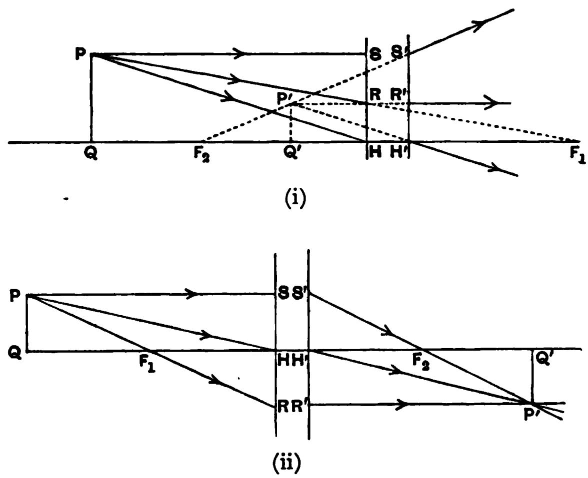


Fig. 35.

Draw the incident ray PF_1R meeting the first unit plane in R. This ray emerges in a direction parallel to the axis, and passes through R' on the second unit plane.

Draw the incident ray PS parallel to the axis; this ray emerges in the direction F_2S'. The two emergent rays intersect in the focus P', conjugate to P.

It is obvious from similar triangles that

$$\left.\begin{aligned} \frac{P'Q'}{PQ} &= \frac{P'Q'}{H'S'} = \frac{F_2Q'}{F_2H'} \\ &= \frac{HR}{PQ} = \frac{F_1H}{F_1Q} \end{aligned}\right\} \quad \text{.....................(V).}$$

Hence $$F_1Q \,.\, F_2Q' = F_1H \,.\, F_2H' \quad \text{..................(VI).}$$

These formulæ hold good in all cases, whatever the positions of Q and Q' with regard to the principal foci and unit points. The positions of these cardinal points with regard to the surfaces of the lens are very varied. They can be found in any given case by formulæ (I) and (II); the leading types are exemplified below (Art. 58).

55. Maxwell's formula for the Elongation*.

If Q and R be two points on the axis, Q' and R' their geometrical foci, the ratio $Q'R'/QR$ has been called the elongation. It is the linear magnification for distances parallel to the axis.

Since $$F_1Q\,.\,F_2Q' = F_1R\,.\,F_2R' = -f^2,$$
we obtain at once
$$Q'R' = -f^2(1/F_1Q - 1/F_1R) = QR\,.\,f^2/F_1Q\,.\,F_1R.$$

Hence $Q'R'/QR = m_Q\,.\,m_R$, where m_Q, m_R are the values of the linear magnification for the points Q and R.

It follows from this that, if PQ be any small straight line inclined to the axis at any angle ϵ, its image is a small straight line $P'Q'$ inclined to the axis at an angle ϵ'.

Since $$\tan\epsilon'/\tan\epsilon = \frac{P'R'}{Q'R'}\Big/\frac{PR}{QR} = m_R/m_Q m_R = 1/m_Q,$$
the ratio $\tan\epsilon' : \tan\epsilon$ is independent of the points P and R chosen (as long as PR is small enough to allow us to suppose that P' is the geometrical focus of P), and the image of the line PQ oblique to the axis is therefore the line $P'Q'$.

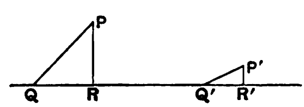


Fig. 36.

Also the equation
$$P'Q'/PQ = P'R'\operatorname{cosec}\epsilon'/PR\operatorname{cosec}\epsilon$$
$$= m_R(\sin^2\epsilon + m_Q^2\cos^2\epsilon)^{\frac{1}{2}}$$
gives the value of the oblique magnification.

56. Paths of rays which do not meet the axis.

Let two fixed planes through the axis at right angles to each other be taken as planes of reference zy, zx. Then we may define any incident ray by means of its direction-cosines referred to the axes of reference, and by the coordinates, referred to these planes, of any point on the ray. Similarly for the emergent ray. First, let any incident ray, whose direction-cosines are $(l, m, 1)$ where l, m are small quantities, whose squares are neglected, intersect the first focal plane in D, and the first unit plane in a point T of coordinates (ξ, η). Then the emergent ray meets the second unit plane in a

* Maxwell, "On the general laws of optical instruments," *Collected Works*, Vol. I. pp. 271—285.

point T' also of coordinates (ξ, η); and if a ray DR be drawn parallel to the axis meeting the first unit plane in R, then the corresponding ray on

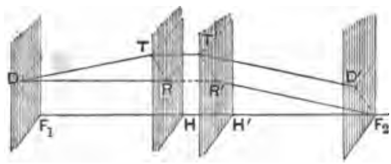


Fig. 37.

emergence is $R'F_2$, and the ray corresponding to DT on emergence is $T'D'$, parallel to $R'F_2$, since the origin D of these two rays is on the first focal plane.

If (ξ_0, η_0) be the coordinates of D or R or R', and $(l', m', 1)$ be the direction-cosines of the emergent ray $T'D'$, then, since to our order of approximation $DT = -f = T'D'$ (the figure being drawn for a convergent lens for which the focal length f is negative), we have by projecting DT and $R'F_2$ on the axes of reference,

$$\xi - \xi_0 = l(-f), \quad \eta - \eta_0 = m(-f),$$

$$-\xi_0 = l'(-f), \quad -\eta_0 = m'(-f);$$

whence
and

$$\left.\begin{aligned} l' - l &= \xi/f = K\xi \\ m' - m &= \eta/f = K\eta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III)}.$$

Again, let the incident ray cut the plane drawn perpendicular to the axis through Q, a point on the axis, in the point P of coordinates (x, y); and let the emergent ray similarly cut the plane through the conjugate focus Q' in the point P' of coordinates (x', y'); then if $QH = z$, $Q'H' = z'$, both measured positively with the light, the coordinates of T or T' are given by the equations,

$$\xi = x + lz = x' + l'z',$$

$$\eta = y + mz = y' + m'z'.$$

Since the point P and the conjugate focus P' lie in a plane through the axis,

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{f - z'}{f} = \frac{f}{f + z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(V)};$$

the latter pair being simply the formulæ of linear magnification.

57. Centre of a Lens.

The determination of the positions of the unit points H and H' is facilitated, in the case of a lens, by the existence of a certain point, known as the *centre* of the lens.

Any ray which passes through the lens without deviation,

must in its passage through the lens cut the axis in a fixed point, independent of the direction of the ray. For the ray must practically pass through a plate, *i.e.* the tangent planes to the spherical surfaces at the points of incidence and emergence are parallel; and the line joining the points of contact of any two parallel tangent planes is known to cut the axis in one of the two centres of similitude of the spheres. The centre of similitude, C, which corresponds to those parts of the spheres that form the surfaces of the lens, is called its centre. To determine its position we have (cf. Figs. 38—42) the equations

$$AC/\rho = BC/\sigma = AB/(\rho - \sigma).$$

This point C is optically conjugate in the two surfaces of the lens to H and H' respectively.

58. Positions of the principal foci and unit points.

Lenses are divided according to the curvatures of their surfaces, as viewed from the outside, into three classes:—double-concave, double-convex, and meniscus.

A meniscus lens is bounded by surfaces, one of which is concave and the other convex to the outside; and is known as concavo-convex or convexo-concave according as the light is first incident on the concave or convex surface.

Plano-concave, or concavo-plane, and plano-convex or convexo-plane, may be included in the first and second classes respectively.

(1) *Double-concave lens.*

If r and s be the numerical radii of the surfaces, then in this lens the symbol σ is $-s$, and ρ is r.

Hence

$$K = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) + \frac{(\mu - 1)^2 t}{\mu r s}.$$

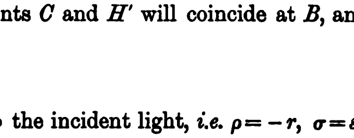


Fig. 38.

The power is therefore positive; also C lies inside the lens, and so do the points H and H'.

For a plano-concave lens, *i.e.* a lens with its plane face to the light, the points C and H' will coincide at B, and $AH = AB/\mu$.

(2) *Double-convex Lens.*

In this lens the first face is convex to the incident light, *i.e.* $\rho = -r$, $\sigma = s$; hence

$$K = -\left\{(\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) - \frac{(\mu - 1)^2 t}{\mu r s}\right\}.$$

The power is negative, except in one case where the thickness of the lens exceeds $\mu/(\mu-1)$ times the sum of the radii.

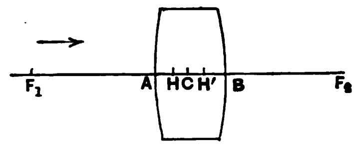


Fig. 39.

The centre C lies within the lens, and between the centres of the surfaces, while, provided that $r+s>t$, as is nearly always the case in practice, H is between A and C, H' between C and B.

If however $r+s=t$, the surfaces are concentric, and C, H, and H' all coincide at the common centre.

If $r+s<t$, then the point H is behind C, while H' is in front of C.

For a convexo-plane lens (which has the advantage of giving a good image at its second focus), C and H coincide at A, and $BH'=BA/\mu$.

Lastly, if $r+s<(\mu-1)t/\mu$, the lens is divergent; both H and F_1 are in front of the lens, H being the more remote, and H' and F_2 are behind the lens, H' being the more remote.

(3) *Meniscus Lens.*

In dealing with this lens we may take both ρ and σ positive, since the power of a lens is not altered by reversing it.

First we see that, if $\sigma<\rho$, the power is essentially negative. In this case the second face is more curved than the first, and the centre lies behind the lens. Also the position of the unit point H is given by $\kappa_2 t/\mu+K.HA=0$, where κ_2 and K are both negative. Hence H lies behind A, and between A and C. So also H' lies behind B and between B and C (Fig. 40).

Since the lens is convergent F_2 is behind H', and F_1 in front of H.

Also from the equation $0=1+\kappa_2 t/\mu+K.F_1A$, the first focus F_1 is in front of A except in one case where $t>\mu\sigma/(\mu-1)$, and the form of the second surface would be a segment greater than a hemisphere.

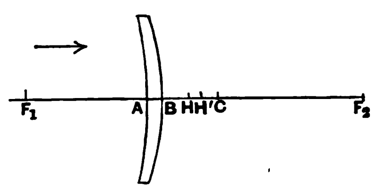


Fig. 40.

Secondly, we see that, if $\sigma>\rho$, the power of the lens is positive or negative as $\sigma-\rho>$ or $<(\mu-1)t/\mu$.

In the first case (Fig. 41) both C and H lie in front of A, and C and H' are in front of B. The centre C lies outside the distance between the centres of the

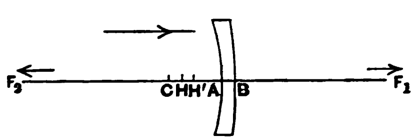


Fig. 41.

surfaces; the points H and H' lie between C and the lens, and in the order

5—2

of Fig. 41 if $\sigma-\rho>t$; the surfaces are concentric when $\sigma-\rho=t$, and then C, H, and H' coincide at the common centre; but when

$$t>\sigma-\rho>(\mu-1)t/\mu,$$

H and H' are in front of C, H' being the more remote.

The principal focus F_1 is always behind A, and the second focus F_2 is in front of H', at considerable distances, since the power of this divergent lens is almost certainly a small quantity.

For the second case (Fig. 42) in which $(\mu-1)t/\mu>\sigma-\rho>0$, the power of the lens is negative; the point C as before is in front of the lens, but the points H and H' are behind it and placed as in the figure. This is evident at once if we notice that C is in front of f_2, the second focus of the first surface A, and hence H, its conjugate, must be behind f_1, the first focus of the same surface.

As the power will be small, the first focus F_1 is a considerable distance in front of H, and F_2 a still greater distance behind the lens.

This shape has no practical importance; it could not exist with power of

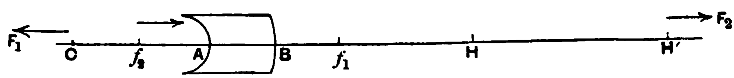


Fig. 42.

any value, except as a very deep-cut microscopic lens, to which the approximations of the theory of geometrical foci are not really applicable.

59. Thin Lens.

If the thickness of a lens can be neglected in comparison with the radii of the surfaces and their algebraic difference, the lens is called thin.

In this case any ray must be considered as entering and leaving the lens at the same point on the common tangent plane to the two surfaces at their vertex A; and if α, α_1, α' be the successive angles of divergence of a ray which before incidence crosses the axis at Q, y the height at which it cuts the lens, we have, as in Art. 49,

$$y=\alpha \,.\, QA,$$

$$\mu\alpha_1-\alpha=(\mu-1)\,y/\rho,$$

$$\alpha'-\mu\alpha_1=(1-\mu)\,y/\sigma.$$

Hence $$\alpha'-\alpha=(\mu-1)\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)y=Ky \quad\text{...........}\quad \text{(III)},$$

whence $$1/Q'A-1/QA=K=1/AF_1=1/F_2A \quad\text{........}\quad \text{(VII)},$$

the second and third equalities being determined by putting Q' and Q respectively at infinity.

Also for the linear magnification, we have

$$l/l' = \alpha'/\alpha = 1 + K.QA \quad \longrightarrow \quad \text{(I)}.$$

Similarly $$l'/l = \alpha/\alpha' = 1 + K.Q'A \quad \longleftarrow \quad \text{(II)}.$$

The principal foci F_1, F_2 are at equal distances $1/K$ from the lens; F_1 lies behind and F_2 in front of the lens, if it be divergent; F_1 is in front and F_2 behind, if the lens be convergent. The focal length f of the lens is equal to $1/K$. The two unit points coincide with the centre of the lens at A.

The image of any object PQ perpendicular to the axis is con-

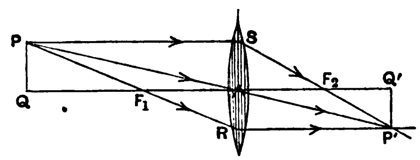


Fig. 43.

structed as before by drawing the ray PF_1R, which emerges parallel to the axis as RP', and the ray PS parallel to the axis, which emerges as SF_2P'; also the ray PA passes through the centre of the lens without deviation.

It is obvious from similar triangles that

$$l'/l = F_2Q'/F_2A = F_1A/F_1Q \text{..................(V)},$$

and hence that

$$F_1Q.F_2Q' = F_1A.F_2A = -f^2 \text{.....................(VI)}.$$

This relation between the position of Q and Q' is given in Newton's *Opticks*, Lond. 1704.

60. Example.

Shew that the unit points of a lens can coincide only when its surfaces are concentric or its thickness is neglected.

Writing down (I) and (II) (Art. 49) we have the positions of the points H, H' given by the equations

$$1 = 1 + \kappa_1 HA + \kappa_2 (HA + t/\mu) + \kappa_1\kappa_2 HA.t/\mu, \quad \longrightarrow$$

$$1 = 1 + \kappa_1 (H'B + t/\mu) + \kappa_2 H'B + \kappa_1\kappa_2 H'B.t/\mu. \quad \longleftarrow$$

Also from the directions in which the distances HA, $H'B$ in these formulæ are measured, the former with the light, and the latter with the light reversed, as indicated by the arrows, it is clear that $HA+H'B=HH'-t$, where HH' is taken positive with the light.

Hence, adding the equations above,

$$0=(\kappa_1+\kappa_2)(HH'-t+t/\mu)+\kappa_1\kappa_2(HH'-t)\,t/\mu,$$

or

$$K\,.\,HH'=(\kappa_1+\kappa_2)\,(1-1/\mu)\,t+\kappa_1\kappa_2 t^2/\mu$$
$$=(\mu-1)^2(\sigma-\rho-t)\,t/\mu\rho\sigma.$$

But if O, O' be the centres of the two spherical surfaces, $OO'=\rho+t-\sigma$.

Hence $$HH'=-(\mu-1)^2\,OO'\,.\,tf/\mu\rho\sigma;$$

i.e. HH' vanishes either when $t=0$, or when O and O' are coincident.

The relative positions of H and H' in the several cases of Art. 58 may be determined by this equation.

61. Example.

Shew that the least distance between an object and its real image formed by a double-convex lens is $f\left\{4+\frac{(\mu-1)^2(r+s-t)t}{\mu rs}\right\}$, *where f is its numerical focal length, and r and s are the radii of the surfaces.*

By considering the positions of the principal foci and unit points relatively to a lens, as shewn in the various cases of Art. 58, and the fact that F_1Q, F_2Q' are always in opposite directions, it is clear that practically only a convergent lens can give a real image of an actual object. It is also obvious that for all practical forms of the lens, H and H' lie between F_1 and F_2.

The distance therefore from the object to the image varies from positive infinity, when the object is at infinity and the image at F_2, to positive infinity again when the object is at F_1 and the image at the further end of the axis; it must therefore have decreased to a minimum value at some point.

Since $QF_1\,.\,F_2Q'=F_1H^2$, it is plain that QQ' is least when

$$QF_1=F_2Q'=F_1H.$$

Hence the minimum distance is found when the object is at L, and the image at L', where F_1 bisects LH, and F_2 bisects $H'L'$. The least distance between the real object and real image is therefore equal to LL', *i.e.* to $4F_1H+HH'$.

The linear magnification for L is -1, and the positions of L and L' are therefore given by

$$-1=1+\kappa_2 t/\mu+K\,.\,LA \longrightarrow$$
$$-1=1+\kappa_1 t/\mu+K\,.\,L'B \longleftarrow$$

where $\kappa_1=-(\mu-1)/r$, $\kappa_2=(1-\mu)/s$, and $K=\kappa_1+\kappa_2+\kappa_1\kappa_2 t/\mu=-1/f$.

Hence by addition of these equations,

$$(LL'-t)/f=4+(\kappa_1+\kappa_2)\,t/\mu,$$

or

$$LL'/f=4+(\kappa_1+\kappa_2)\,t/\mu-t\,(\kappa_1+\kappa_2+\kappa_1\kappa_2 t/\mu)$$
$$=4+(\mu-1)^2(r+s-t)\,t/\mu rs.$$

If the restriction that Q and Q' be real foci be removed, the distance QQ' must be considered as changing to negative infinity as Q passes F_1; it will then increase through zero to HH', and decrease again through zero to negative infinity, when Q has reached the further end of the axis and Q' is at F_2 again.

For a divergent lens the distance QQ' will decrease from positive infinity to HH' and then increase to positive infinity again. In the case of the divergent meniscus HH' may possibly be negative, and the distance QQ' may have vanished twice. After Q passes F_1 the distance QQ' will increase from negative infinity to $-LL'$, and then decrease to negative infinity, changing suddenly to positive infinity as Q' reaches F_2.

62. Determination of the focal length of a lens.

If the radii of the surfaces of a lens can be measured by means of a spherometer (cf. Ex. 1), and if the index of refraction of the lens and its thickness be also known, the focal length and the positions of the cardinal points can at once be determined, as shewn in Art. 58.

When the focal length is found by direct optical experiment, the lens, an origin of light, and a screen to receive the image, are fixed on a graduated axis, called an optical bench. The experiments are practically restricted to real images, unless further apparatus is introduced. As in every case but one a lens must be convergent to produce a real image of a real object, the focal length and the cardinal points of a divergent lens must be found by combining it with a known convergent lens, so that the combination is convergent. The positions of the cardinal points of the combination and of those of the convergent lens must be known to deduce the focal length and the positions of the cardinal points of the divergent lens (cf. Art. 68).

First, if a pencil of parallel rays can be made to pass through the lens in both directions, the positions of the principal foci can be at once determined. The focal length f is then found by measuring the distances QF_1, F_2Q' of any object and its image from the principal foci. The positions of the unit points are then determined by measuring f inwards from F_1 and F_2.

Secondly, the focal length of a lens can be found approximately by measuring the minimum distance between a real object and a real image. The image is then equal to the object, but inverted, and the minimum distance is $4f$, if the thickness be entirely neglected. If however the thickness be small compared with the

radii r, s of the surfaces, and if the lens have a sharp edge, then, y being the radius of aperture of the lens, we have by Newton's formula for curvature, $y^2/2r + y^2/2s = t$, approximately, and the value of the minimum distance given in Art. 61 is approximately

$$4f\left\{1+\frac{(\mu-1)^2}{2\mu}\frac{t^2}{y^2}\right\},$$

where terms of order t^2/rs are neglected in the bracket.

Thirdly, the focal length and the positions of the principal foci may be determined by measuring the distances between three origins of light on the axis, and the distances between their images.

Let P, Q, R be three points in order on the axis, and P', Q', R' their (real) images. Then P, Q, R must lie in front of F_1, and P', Q', R' must lie behind F_2.

Let $x_1 = PQ$, $x_2 = QR$, $x_3 = PR$, $x_1' = P'Q'$, $x_2' = Q'R'$ and $x_3' = P'R'$; and let $\xi = QF_1$, $\xi' = F_2Q'$.

Then we have the equations

$$(\xi + x_1)(\xi' - x_1') = \xi\xi' = (\xi - x_2)(\xi' + x_2') = f^2;$$

from which we deduce

$$\xi/x_1x_2x_3' = \xi'/x_1'x_2'x_3 = 1/(x_1x_2' - x_2x_1').$$

Thus, if the positions of Q and Q' be noted, those of F_1 and F_2 are determined, and the focal length is

$$(x_1x_2x_3x_1'x_2'x_3')^{\frac{1}{2}}/(x_1x_2' - x_2x_1').$$

Another method of determining f is given in Example 19.

Lastly, if we can measure by a micrometer the linear magnification, the focal length can be determined by measuring the difference of the distances between an object and its image in two cases, and the linear magnifications.

If m be the (numerical) linear magnification, we have

$$Q_1Q_1' = f/m_1 + F_1F_2 + m_1f,$$
$$Q_2Q_2' = f/m_2 + F_1F_2 + m_2f.$$

Hence $Q_1Q_1' - Q_2Q_2' = f(m_1 - m_2)(m_1m_2 - 1)/m_1m_2$;

and the positions of F_1, F_2 are then given by

$$Q_1F_1 = f/m_1, \text{ and } F_2Q_1' = m_1f.$$

Other methods are given in Czapski's *Theorie der optischen Instrumente*, Chap. IX.

EXAMPLES.

1. Equal pegs are fixed at the corners of an equilateral triangle at right angles to its plane; a peg, moveable by a screw, is placed at the centre of the triangle. Explain how the curvature of a surface may be measured by this machine.

If the screw peg be higher than the others by 0·765 mm., and one side of the triangle be 107·1 mm., shew that the radius of the surface with which the pegs are all in contact is 2499·3825 mm.

2. Prove that when a thin lens is used to form a real image of a bright object there are also formed by internal reflections a series of fainter images nearer the lens than the principal one, and that the distances of these images from the lens form a harmonical progression.

3. Deduce from Helmholtz's theorem that the effect of any system of n refracting media bounded by concentric spherical shells, on a small pencil incident directly, is equivalent to that of a thin lens at the centre; and shew that the power of the thin lens is $2\sum_{r=1}^{r=n}(1/\mu_r - 1/\mu_{r-1})/R_r$, where R_r is the radius of the surface of separation of the media of indices μ_{r-1} and μ_r, and the system is placed in air.

4. A spherical shell of radii r, s made of glass of refractive index μ has its cavity filled with liquid of index μ' greater than μ, and is used by a person, whose greatest distance of distinct vision is λ, to observe the virtual image of a small object. Shew that, if the focal length be not less than the outer radius r, the greatest possible linear magnification occurs with the eye close to the glass, and is

$$1+2(\lambda-r)\left(\frac{\mu-1}{\mu r}+\frac{\mu'-\mu}{\mu\mu' s}\right).$$

5. A small pencil is refracted directly through a hemisphere of radius r, the light being internally reflected $2n$ times within the lens; shew that the effect is the same as if the light had been refracted through a lens of focal length $(-)^n r/(2n-\mu+1)$, the distances of whose foci from the centre of the hemisphere are respectively $\mu r/(2n-\mu+1)$ and $r/\mu(2n-\mu+1)$ on opposite sides.

6. A thin lens of index μ, the radii of whose surfaces are ρ and σ, is placed at distance d in front of a plane mirror, the medium between the second surface of the lens and the mirror being of index μ'.

Shew that the effect of two refractions through the lens with an intermediate reflection at the plane is exactly equivalent to a reflection at a spherical mirror whose vertex is at distance $1\Big/\left(\frac{\mu'}{d}+\frac{\mu-1}{\rho}+\frac{\mu'-\mu}{\sigma}\right)$, and centre at distance $1\Big/\left(\frac{\mu-1}{\rho}+\frac{\mu'-\mu}{\sigma}\right)$ behind the lens.

7. A small object is at distance a from the eye, and subtends directly an angle 2α. Shew that to form an image at distance b from the eye and subtending an angle 2β, a thin lens of focal length $\dfrac{ab(a-b)\alpha\beta}{(a\alpha-b\beta)^2}$ must be placed at distance $\dfrac{ab(\alpha-\beta)}{a\alpha-b\beta}$ from the eye.

8. Shew that, if Q_1, Q_2, Q_3, Q_4 be any four points on the axis of a lens, Q_1', Q_2', Q_3', Q_4' their geometrical foci, the anharmonic ratios of the two ranges which they form are equal.

Deduce that $H'Q'/HQ=F_2Q'/HF_1=H'F_2/F_1Q$.

9. A double-convex lens is formed by two equal paraboloidal surfaces cut off by planes through the focus perpendicular to the axis. Prove that for rays passing in the neighbourhood of the axis, the minimum distance between a bright point and its image is $2a(\mu+1)/(\mu-1)$, where $4a$ is the latus rectum of either of the generating parabolas.

10. A double-convex lens is such that the thickness exceeds the sum of the radii, also μ times the distance between the centres exceeds the thickness by a positive quantity c; shew that the lens is divergent, and that the distances of the unit points from the two surfaces are respectively rt/c and st/c, where r and s are the radii of the surfaces, t the thickness.

Determine also the positions of the principal foci.

11. Shew that, if the origin of a pencil of rays passing through a divergent lens lie between the first unit point and the lens, the angle of divergence of the pencil is decreased.

12. The centre of a lens lies between the unit points if the lens be double-concave or double-convex; otherwise outside the distance between those points.

13. Shew that a thick lens can be turned about a certain point on its axis so that the images of all objects are unaltered. If the thickness of the lens be small compared with $(\sigma-\rho)$, this point is on the same side of the middle point of the thickness as the centre of the lens, but at $1/\mu$ times its distance approximately.

14. Shew that there are two points on the axis of a thick lens such that each is its own image, and that these points are real if $c\{c+4\mu\rho\sigma/(\mu-1)^2t\}$ be positive, where c is the distance from the centre of the first surface to the centre of the second surface measured in the same direction as the thickness t.

Shew that the linear magnifications for these two points are reciprocals, and if O denote either of the points, m the linear magnification for that point, and Q, Q' be a pair of conjugate foci, $m/OQ'-1/mOQ=1/F_1H$.

15. Shew that a single thin lens can be found to give the same geometrical foci of all points on the axis as a given thick lens, if the distance between the centres of the surfaces be $4\mu\rho\sigma/(\mu-1)^2 t$.

16. A convergent lens is placed so that its second principal focus coincides with the centre of a concave spherical mirror on the same axis. Shew that the image of any small object, formed by refraction through the lens in both directions with an intermediate reflection at the mirror, is identical in position and magnitude with the image formed by a single reflection in a certain plane mirror, and determine the position of this mirror.

17. A convergent lens is placed so that the image of an object viewed through it is upright and at distance λ from the eye. Shew that, to obtain the greatest *angular* magnification, the distance from the first principal focus to the object, measured towards the lens, is $2f^2/(d+\lambda)$, and from the second principal focus to the eye, measured away from the lens, is $\frac{1}{2}(\lambda-d)$, where d is the distance between the principal foci.

18. The least distance between a point and its image formed by a thin convergent lens is a; the least distance for a sphere of refractive index $\frac{4}{3}$ is b. A hollow is made inside the sphere in which the lens is placed, their centres coinciding, and the least distance between an object and its image is now c. Shew that the refractive index of the lens is

$$\{12bc-16a(b-c)\}/\{9bc-16a(b-c)\}.$$

19. The focal length of a thick convergent lens may be found practically as follows. Taking a fixed origin of light and a fixed screen, measure the distance c between the two positions of the lens in which it forms an image of the light on the screen. Increase the distance between the light and the screen by h, and measure the similar distance c', then if $2s=h+c+c'$,

$$f=\{-s(s-h)(s-c)(s-c')\}^{\frac{1}{2}}/h.$$

20. Three points on the axis of a lens are separated by intervals x_1 and x_2 between the first and second, and the second and third respectively. Also $x_3=x_1+x_2$. The corresponding intervals between their images are y_1, y_2 and y_3. Shew that the focal length is $\pm\{x_1x_2x_3y_1y_2y_3\}^{\frac{1}{2}}/(x_1y_2-x_2y_1)$, and if Q be the middle point of the three, Q' its image, the distance between the unit points of the lens is

$$QQ'-\{\sqrt{x_1x_2y_3}\mp\sqrt{y_1y_2x_3}\}^2/(x_1y_2-x_2y_1),$$

where in each expression the upper sign is given to the ambiguity if the linear magnification for Q be positive.

21. The minimum deviation θ of a ray, which passes at the edge of a double-convex lens, whose thickness is $2t$, and the radii of whose surfaces are r and s, is given by

$$(\mu+1)\tan\tfrac{1}{4}\theta=\left\{\frac{(r-t)(s-t)}{t(r+s-t)}\right\}^{\frac{1}{2}}-\left\{\frac{rs}{t(r+s-t)}-\mu^2\right\}^{\frac{1}{2}}.$$

22. Two spherical mirrors of focal lengths f_1 and f_2 are placed on the same axis, and the distance between their principal foci is c. Shew that for $2n$ reflections the effect is the same as if the pencil passed through a lens of focal length $\frac{f_1 f_2}{c} \sinh \gamma \operatorname{cosech} n\gamma$, whose first principal focus is at a distance

$$f_1 \{f_1 \sinh n\gamma + f_2 \sinh (n-1)\gamma\}/c \sinh n\gamma$$

from the focus of the first surface.

For $2n+1$ reflections, the effect is the same as if the pencil were reflected at a spherical surface of focal length $f_1 f_2 \sinh \gamma / \{f_1 \sinh n\gamma + f_2 \sinh (n+1)\gamma\}$, whose focus is at distance $cf_1 \sinh n\gamma / \{f_1 \sinh n\gamma + f_2 \sinh (n+1)\gamma\}$ from the focus of the first surface, and γ is given by the equation

$$2f_1 f_2 \cosh \gamma = c^2 - f_1^2 - f_2^2.$$

CHAPTER V.

COAXIAL REFRACTING SURFACES.

63. THE determination of the cardinal points of any system of coaxial refracting surfaces, and of the effect of the system on a small pencil directly incident by means of these points, was first made by Gauss*.

I have not followed this method, which is entirely analytical, but have preferred to make use of a more geometrical method, known as Cotes's theorem, and given in Smith's *Compleat System of Opticks*†, Cambridge, 1738.

This theorem, with the help of Helmholtz's formula, is sufficient to determine all the properties of a symmetrical optical instrument for pencils incident directly.

64. System of Thin Lenses.

Let a system of n thin lenses be arranged on the same axis, which meets them successively in the points $A_1, A_2 \dots A_n$; and let their powers be $\kappa_1, \kappa_2 \dots \kappa_n$.

Let a ray from a point Q on the axis cross them at heights $y_1, y_2, \dots\dots y_n$, and let the successive angles of divergence of the ray from the axis be $\alpha_0, \alpha_1 \dots \alpha_n$.

In the standard case, where all the lenses are divergent, these quantities y and α will be positive throughout, but if any of the lenses be convergent α may become negative, and further, if the ray in passing from one lens to the next cross the axis between them, the corresponding α and the height y on the second lens, as well as those that follow, must be considered negative (cf. Art. 46).

* Gauss, "Dioptrische Untersuchungen," *Werke*, Bd. v. p. 247.

† Smith, *Opticks*, Vol. I. p. 111, Prop. I. and corollaries.

With the conventions of that article we can form the equations

$$\left.\begin{aligned} y_1 &= QA_1 . \alpha_0 \\ \alpha_1 - \alpha_0 &= \kappa_1 y_1 \\ y_2 - y_1 &= A_1A_2 . \alpha_1 \\ \alpha_2 - \alpha_1 &= \kappa_2 y_2 \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ y_n - y_{n-1} &= A_{n-1}A_n . \alpha_{n-1} \\ \alpha_n - \alpha_{n-1} &= \kappa_n y_n \end{aligned}\right\} \qquad \text{(i).}$$

Plainly, if we substitute for y_1/α_0 from the first of these

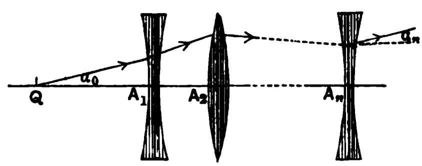


Fig. 44.

equations in the second and third, then for α_1/α_0 in the third and fourth, and so on, we finally obtain α_n/α_0 as a function of the powers and distances apart of the lenses, into which QA_1 enters only linearly.

This relation between α_n and α_0 may be written in the form

$$\alpha_n/\alpha_0 = K . QA_1 + \frac{\partial K}{\partial \kappa_1}, \quad \longrightarrow \qquad \text{(I),}$$

where K contains only the powers and distances apart of the lenses, and $\dfrac{\partial K}{\partial \kappa_1}$ is written for the coefficient of κ_1 in K.

For if we substitute from the third of equations (i) in the fourth for y_2, and then in the next equation for α_2, and so on, we obtain an equation which may be written as

$$\begin{aligned} \alpha_n &= py_1 + q\alpha_1, \\ &= (p + q\kappa_1)\, y_1 + q\alpha_0, \text{ from the second equation,} \\ &= \{(p + q\kappa_1)\, QA_1 + q\}\, \alpha_0, \text{ from the first equation,} \end{aligned}$$

where p and q do not contain κ_1.

The relation is therefore of the form given above.

Moreover, by the use of Helmholtz's formula (Art. 47), this equation may be replaced by

$$l/l' = K \cdot QA_1 + \frac{\partial K}{\partial \kappa_1},$$

where l and l' are respectively the linear dimensions of a small object perpendicular to the axis at Q, and of its final image at Q'.

Hence it follows that, *excluding for the present the case in which K is zero* (cf. Art. 72), the linear magnification varies inversely as the distance of the object from a fixed point, and takes every value once, and once only.

The position of the first unit point H is given by putting $l = l'$; and that of the first principal focus F_1 by putting α_n zero.

Hence
$$1 = K \cdot HA_1 + \frac{\partial K}{\partial \kappa_1},$$
$$0 = K \cdot F_1A_1 + \frac{\partial K}{\partial \kappa_1}.$$

Therefore
$$HF_1 = 1/K.$$

This quantity K is called the *power* of the system, and the distance HF_1, *taken positive when measured onwards with the light, is its focal length.*

In the same way by reversing the path of the ray, we obtain an equation of the form

$$\alpha_0/\alpha_n = K' \cdot Q'A_n + \frac{\partial K'}{\partial \kappa_n} = l'/l, \qquad \text{(II)}.$$

The linear magnification is therefore directly proportional to the distance of the image from a fixed point; and we can determine uniquely the positions of the second unit point H' and of the second principal focus F_2.

The geometrical constructions given in Art. 54 hold good in all respects; and we therefore obtain

(i) $HF_1 = F_2H'$, *i.e.* $K' = K$,

(ii) $l'/l = F_2Q'/F_2H' = F_1H/F_1Q$,

(iii) $F_1Q \cdot F_2Q' = F_1H \cdot F_2H' = -(1/K)^2 = -f^2$,

(iv) $\alpha_n - \alpha_0 = Ky$,

where y is the distance from the axis at which the incident ray meets the first unit plane of the system.

The final path of a ray which does not cut the axis can be determined in terms of its initial path exactly as in Art. 56.

65. Cotes's Formulæ.

The value of α_n/α_0 may be expressed in the following form, which involves only simple geometrical distances:

$$\begin{aligned}\alpha_n/\alpha_0 = 1 &+ \Sigma\kappa_r QA_r + \Sigma\kappa_r\kappa_s QA_r . A_rA_s \\ &+ \Sigma\kappa_r\kappa_s\kappa_t QA_r . A_rA_s . A_sA_t \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \kappa_1\kappa_2 \ldots\ldots \kappa_n QA_1 . A_1A_2 . \ldots . A_{n-1}A_n \longrightarrow \quad (\text{I}).\end{aligned}$$

In this summation all possible combinations of the powers, one, two, three...... n at a time are taken, and the coefficient of any particular combination is the continued product of the distance from Q to the first lens of that combination, the distance from that lens to the second of the combination, the distance from the second to the third, and so on, *always moving onwards from Q with the light.*

It is essential that in any product of intervals as A_rA_s, A_sA_t, ... no part of the axis be traversed twice in the same direction.

The formula is proved by induction.

From the first and second of equations (i) we deduce

$$\alpha_1/\alpha_0 = 1 + \kappa_1 QA_1.$$

Substitute in the third,

$$\begin{aligned} y_2/\alpha_0 &= QA_1 + (1 + \kappa_1 QA_1)\, A_1A_2 \\ &= QA_2 + \kappa_1 QA_1 . A_1A_2. \end{aligned}$$

Substituting from both of these in the fourth,

$$\alpha_2/\alpha_0 = 1 + \kappa_1 QA_1 + \kappa_2 QA_2 + \kappa_1\kappa_2 QA_1 . A_1A_2.$$

Hence the formula holds for the values $n = 1$ or 2.

To prove it generally, assume that the form given is true for $(n-2)$ and $(n-1)$ lenses.

Then from the last three equations, (i) Art. 64, we deduce

$$(\alpha_n - \alpha_{n-1})/\kappa_n - (\alpha_{n-1} - \alpha_{n-2})/\kappa_{n-1} = y_n - y_{n-1} = A_{n-1}A_n . \alpha_{n-1}.$$

Hence $(\alpha_n - \alpha_{n-1})/\alpha_0 = \dfrac{\kappa_n}{\kappa_{n-1}} (\alpha_{n-1} - \alpha_{n-2})/\alpha_0 + \kappa_n A_{n-1}A_n . \alpha_{n-1}/\alpha_0.$

Now the terms in which the form assumed for a_{n-1}/a_0 exceeds that for a_{n-2}/a_0 are only those which involve κ_{n-1} as a factor, and therefore $(a_{n-1} - a_{n-2})/a_0$ may be written as

$$\kappa_{n-1}\{QA_{n-1} + \Sigma\kappa_r\kappa_s\ldots\kappa_z QA_r . A_rA_s \ldots . A_zA_{n-1}\},$$

where the summation extends to all possible combinations of the powers and to all values of the suffixes $r, s \ldots z$ from 1 to $(n-2)$ inclusive.

Hence

$$(a_n - a_{n-1})/a_0 = \kappa_n\{QA_{n-1} + \sum_1^{n-2} \kappa_r\kappa_s\ldots\kappa_z QA_r . A_rA_s \ldots . A_zA_{n-1}\}$$

$$+ \kappa_n A_{n-1}A_n\{1 + \sum_1^{n-1} \kappa_r\kappa_s\ldots\kappa_z QA_r . A_rA_s \ldots . A_tA_z\},$$

$$= \kappa_n\{QA_n + \sum_1^{n-1} \kappa_r\kappa_s\ldots\kappa_z QA_r . A_rA_s \ldots . A_zA_n\},$$

since $$A_zA_{n-1} + A_{n-1}A_n \equiv A_zA_n.$$

This is precisely the same form as the previous difference with n in place of $(n-1)$; and therefore the form given for a_n/a_0 holds for any number of lenses.

It must be remembered that in this formula QA is an algebraic quantity, to be reckoned positive for divergent pencils; and if the pencil be convergent towards Q before incidence, *i.e.* if Q be behind any of the lenses, the distances from Q to those lenses must be reckoned negative. Also the power κ is positive for a divergent lens, negative for a convergent lens.

By retracing the path of the ray from the geometrical focus Q' we obtain the corresponding formula

$$\begin{aligned} a_0/a_n = 1 &+ \Sigma\kappa_r Q'A_r + \Sigma\kappa_s\kappa_r Q'A_s . A_sA_r \\ &+ \Sigma\kappa_t\kappa_s\kappa_r Q'A_t . A_tA_s . A_sA_r \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \kappa_n\kappa_{n-1}\ldots\kappa_1 Q'A_n . A_nA_{n-1} \ldots . A_2A_1 \quad \longleftarrow \quad \text{(II)}. \end{aligned}$$

The light now traverses the system in the opposite direction, and meets the lenses in reverse order, while $Q'A$ *is positive if measured with the light reversed.*

66. Power.

The power K of the system is the coefficient of QA_1 in formula (I), when the substitutions $QA_r \equiv QA_1 + A_1A_r$ are performed. From the form of (I) it is obvious that

$$K = \Sigma\kappa_r + \Sigma\kappa_r\kappa_s A_rA_s + \Sigma\kappa_r\kappa_s\kappa_t A_rA_s \,.\, A_sA_t$$
$$+ \ldots + \kappa_1\kappa_2\ldots\kappa_n A_1A_2 \,.\, A_2A_3 \,.\, \ldots \,.\, A_{n-1}A_n \ldots\text{(III)},$$

where every possible combination of the powers of the n lenses is taken, and the coefficient of any combination is the product of the distance from the first lens of that combination to the second, the distance from the second to the third, and so on, till the combination is exhausted.

It is clear from this form that the power of the system is the same whether the light traverses the system in the direction A_1A_n or in the direction A_nA_1 (cf. § 64).

67. It is obvious from the equations

$$\left.\begin{aligned} y_1 &= ua_0 \\ a_1 - a_0 &= \kappa_1 y_1 \\ y_2 - y_1 &= a_1 a_1 \\ &\cdots\cdots \\ a_n - a_{n-1} &= \kappa_n y_n \end{aligned}\right\} \ldots\ldots\ldots\ldots\text{(i)},$$

where $u = QA_1$, $a_1 = A_1A_2$, &c., that a_n/a_0 is the numerator of the last convergent to the continued fraction $u + \frac{1}{\kappa_1 +}\,\frac{1}{a_1 +}\,\frac{1}{\kappa_2 +}\,\frac{1}{a_2 +}\cdots\frac{1}{\kappa_n}$.

By a theorem in continued fractions* this is also the numerator of the last convergent to the continued fraction

$$\kappa_n + \frac{1}{a_{n-1} +}\cdots\frac{1}{\kappa_1 +}\,\frac{1}{u}.$$

Therefore $$a_n/a_0 = uK + \frac{\partial K}{\partial \kappa_1}\ldots\ldots\ldots\ldots\text{(I)},$$

where K = numerator of last convergent to $\kappa_n + \frac{1}{a_{n-1} +}\cdots\frac{1}{\kappa_1}$.

$\frac{\partial K}{\partial \kappa_1}$ = numerator of last convergent but one to K = the coefficient of κ_1 in K.

Applying the same theorem again, we have

K = numerator of last convergent to $\kappa_1 + \frac{1}{a_1 +}\cdots\frac{1}{\kappa_n}$.

$\frac{\partial K}{\partial \kappa_1}$ = denominator " " " "

$\frac{\partial K}{\partial \kappa_n}$ = numerator of last convergent but one " "

$\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n}$ = denominator " " " "

* Chrystal's *Algebra*, Vol. II. p. 406.

Hence $$\frac{\partial K}{\partial \kappa_1}\frac{\partial K}{\partial \kappa_n} - K\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} \equiv 1.$$

Multiplying together equations I, II, Art. 64, we obtain the relation between the distance u of the object in front of the first lens and the distance v of the image behind the last lens in the form

$$\left(uK + \frac{\partial K}{\partial \kappa_1}\right)\left(vK + \frac{\partial K}{\partial \kappa_n}\right) = 1,$$

or, making use of the above identity, in the form

$$uvK + u\frac{\partial K}{\partial \kappa_n} + v\frac{\partial K}{\partial \kappa_1} + \frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} = 0.$$

But in any special case this form of the relation between conjugate foci is more easily obtained by using the formula for Apparent Distance given below, Art. 86.

68. System of Thick Lenses on the same Axis.

Let a ray pass through a given coaxial system of thick lenses. Since the deviation produced in a ray by a lens is κy, where κ is its power, y the height at which the ray meets the unit planes (cf. III, Art. 51), we can form exactly the same equations (i) as in Art. 64, provided that in those equations we write QH_1 for QA_1, and for the distance between consecutive lenses, *e.g.* A_1A_2, we write $H_1'H_2$, the algebraic distance from the second unit plane of the first lens to the first unit plane of the second.

The formulæ (I, II, and III, Arts. 65, 66) for the magnification and for the power K of the entire system will remain as before if we understand by the distance between any two lenses the sum of these algebraic distances from the second unit plane of a lens to the first unit plane of the consecutive lens.

The positions of the cardinal points F_1, F_2, H and H' of the combination of lenses are found in the same way as when the lenses are thin by equating the values of α_n/α_0 or α_0/α_n, as the case may be, to zero or unity.

Another method of forming (I) is given in Art. 75, since the n lenses may be treated as a case of $2n$ coaxial refracting surfaces.

69. We may also determine the positions of the principal foci and the power of a system of lenses by the use of the principal foci of the lenses, as follows.

Let a system of n lenses (thick or thin) of (algebraic) focal lengths $f_1, f_2 \ldots f_n$ be arranged on the same axis. Let c_1, c_2, c_3, c_{n-1} be the respective distances from the second focus of one lens to the first focus of the consecutive lens, taken positive if measured onwards with the light.

Let u be the distance of a point Q on the axis in front of the first focus ${}_1F_1$ of the first lens; let $u_1, u_2 \ldots u_{n-1}$ be the distances of its successive geometrical foci in front of the first foci of the succeeding lenses in order; and let v be the distance of the final geometrical focus behind the second focus ${}_nF_2$ of the last lens.

We can form the equations

$$\left.\begin{aligned} u(c_1 - u_1) &= f_1^2 \\ u_1(c_2 - u_2) &= f_2^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots &\ldots\ldots \\ u_{n-2}(c_{n-1} - u_{n-1}) &= f^2_{n-1} \\ u_{n-1} v &= f_n^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Hence u_r can be expressed as a continued fraction

$$c_r - \frac{f_r^2}{c_{r-1} -} \cdots \frac{f_1^2}{u}.$$

If we denote this by p_r/q_r, then

$$u_{r+1} = \frac{p_{r+1}}{q_{r+1}} = c_{r+1} - \frac{f^2_{r+1} q_r}{p_r} = \frac{c_{r+1} p_r - f^2_{r+1} q_r}{p_r},$$

and therefore $q_{r+1} = p_r$, it being understood that the numerators and denominators of the convergents to the continued fraction are calculated separately, and that no common factors are ever removed.

Now the linear magnification $= \dfrac{f_1 f_2}{u\, u_1} \cdots \dfrac{f_n}{u_{n-1}}$ .... (ii)

$$= (f_1 f_2 \ldots f_n) \Big/ \left(u\, \frac{p_1}{u}\, \frac{p_2}{q_2} \cdots \frac{p_{n-1}}{q_{n-1}}\right)$$

$$= (f_1 f_2 \ldots f_n)/p_{n-1},$$

$$\text{or } l/l' = \frac{\left[\text{numerator of last convergent to } \left(c_{n-1} - \dfrac{f^2_{n-1}}{c_{n-2} -} \cdots \dfrac{f_1^2}{u}\right)\right]}{f_1 f_2 \ldots f_n},$$

$$= \left(u\varpi - f_1^2 \frac{\partial \varpi}{\partial c_1}\right) \Big/ (f_1 f_2 \ldots f_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

where ϖ is the numerator of the last convergent to

$$c_{n-1} - \frac{f^2_{n-1}}{c_{n-2} -} \cdots \frac{f_2^2}{c_1},$$

which is also the numerator of the last convergent to

$$c_1 - \frac{f_2^2}{c_2 -} \cdots \frac{f^2_{n-1}}{c_{n-1}}.$$

But this expression for l/l' must be equal to $K.QF_1$ or QF_1/HF_1, where F_1 is the first focus and K the power of the entire system. Comparing this form with (iii) we see that the first focus of the entire system is at a distance $\frac{f_1^2}{\varpi}\frac{\partial\varpi}{\partial c_1}$ in front of ${}_1F_1$; the second focus of the entire system is similarly a distance $\frac{f_n^2}{\varpi}\frac{\partial\varpi}{\partial c_{n-1}}$ behind ${}_nF_2$; and the power of the system is $\varpi/(f_1 f_2 \ldots f_n)$.

70. Equivalent Lens.

A thin lens of the same power as a given system of lenses will, if placed at the first unit point of the system, produce in any ray the same deviation as the system.

This lens is called the *Equivalent Lens.*

The deviation produced by the system is given (iv. Art. 64) by the equation

$$\alpha_n - \alpha_0 = Ky,$$

where y is the distance from the axis at which the ray meets the unit planes of the system.

This equation also holds for a thin lens (Art. 59), but then the unit planes coincide at the lens.

Hence to produce the same deviation in any ray the thin lens must be placed at the first unit point, and its power must be K.

This proposition is even more obvious from the geometrical constructions (Arts. 54, 59), which shew also that the system and the lens produce identically equal images of any object, separated however by the constant distance HH' between the unit planes of the system.

71. Examples.

(i) *Shew that for two thin convergent lenses of numerical focal lengths* f_1 *and* f_2 *on the same axis at a distance* a *apart greater than* (f_1+f_2), *the minimum distance between a real object and a real image is* $a^2/(a-f_1-f_2)$.

Shew however that when a *is less than both* $2f_1$ *and* $2f_2$ *the minimum distance is* $(4f_1f_2-a^2)/(f_1+f_2-a)$.

It was shewn in Art. 61 that the distance between two conjugate foci is a minimum when they are at the unit points H, H', and also when they are at the points of negative unit magnification.

By Cotes's formula (I) we have

$$a_2/a_0=1-u/f_1-(u+a)/f_2+ua/f_1f_2$$
$$=1-a/f_2+u(a-f_1-f_2)/f_1f_2.$$

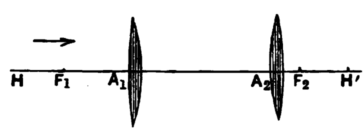


FIG. 45.

Hence in the first case the two lenses form a divergent system; and putting $a_2=a_0$ we see that H lies at a distance $af_1/(a-f_1-f_2)$ in front of A_1. Similarly H' is at distance $af_2/(a-f_1-f_2)$ behind A_2.

The principal foci F_1 and F_2 lie between H and H', and hence in this case the distance between a real object and its real image is a minimum when they are respectively at H and H'.

The minimum distance is therefore equal to

$$af_1/(a-f_1-f_2)+a+af_2/(a-f_1-f_2)=a^2/(a-f_1-f_2).$$

If however $a<f_1+f_2$ the system is convergent, H and H' lie within A_1A_2, and the minimum distance between a real object and its real image occurs when the object and image are at the points L, L' of magnification -1, provided these lie outside A_1A_2.

Their positions are given by the equations

$$-1=1-a/f_2-LA_1(f_1+f_2-a)/f_1f_2, \longrightarrow$$
$$-1=1-a/f_1-L'A_2(f_1+f_2-a)/f_1f_2. \longleftarrow$$

Hence adding these equations

$$(LL'-a)(f_1+f_2-a)/f_1f_2=4-a(f_1+f_2)/f_1f_2,$$

or
$$LL'=(4f_1f_2-a^2)/(f_1+f_2-a),$$

and the light passes really through L and L', provided LA_1, $L'A_2$, as given above, are both positive, *i.e.* provided a be less than both $2f_1$ and $2f_2$.

(ii) *Shew that as far as the position of the image is concerned any system of lenses can be replaced by a single thin lens if one or other of two conditions be secured. In the case of three thin lenses of focal lengths* f_1, f_2, f_3 *at distances* d_3, d_1 *apart, shew that the conditions are*

$$f_1d_1^2+f_2d_2^2+f_3d_3^2+d_1d_2d_3=0 \text{ or } =4f_1f_2f_3;$$

where $d_2=d_3+d_1$.

Let Q, Q' be any pair of conjugate foci in a refracting system, R, R' another pair. Then

$$F_1Q \,.\, F_2Q' = (F_1R + RQ)\,(F_2R' + R'Q') = F_1R \,.\, F_2R'.$$

Hence $F_2R'/R'Q' + F_1R/RQ + 1 = 0$; or if m_R be the linear magnification for the point R, which is equal to F_2R'/F_2H' and to F_1H/F_1R, we deduce

$$m_R/R'Q' - m_R^{-1}/RQ = 1/H'F_2 = 1/F_1H.$$

For a thin lens placed at R we have the equation

$$1/RQ' - 1/RQ = 1/f.$$

Hence if every pair of conjugate foci are the same for the thin lens at R as for the system we must have $m_R = \pm 1$, *i.e.* R, R' must be the unit points H, H', or the inverse unit points L, L', and also (i) if $m_R = +1$, the points H, H' must coincide, or (ii) if $m_R = -1$, the points L, L' must coincide.

In the first case, when H and H' coincide, the thin lens is the equivalent lens of the system, and produces exactly the same image in magnitude and position as the system. But in the second case, when L and L' coincide, the thin lens has focal length opposite to that of the system, and produces an image at the same point as the system, but exactly inverted.

When the system consists of three thin lenses A_1, A_2, A_3 of powers κ_1, κ_2, κ_3, separated by intervals d_3, d_1 and d_2 $(\equiv d_3 + d_1)$, we have by Cotes's formulæ for any point Q,

$$\begin{aligned} 1/m_Q = 1 &+ \kappa_1 QA_1 + \kappa_2 QA_2 + \kappa_3 QA_3 \\ &+ \kappa_1\kappa_2 QA_1 \,.\, d_3 + \kappa_1\kappa_3 QA_1 \,.\, d_2 + \kappa_2\kappa_3 QA_2 \,.\, d_1 \\ &+ \kappa_1\kappa_2\kappa_3 QA_1 \,.\, d_3 \,.\, d_1. \quad \longrightarrow \end{aligned}$$

$$\begin{aligned} m_Q = 1 &+ \kappa_1 Q'A_1 + \kappa_2 Q'A_2 + \kappa_3 Q'A_3 \\ &+ \kappa_1\kappa_2 Q'A_2 \,.\, d_3 + \kappa_1\kappa_3 Q'A_3 \,.\, d_2 + \kappa_2\kappa_3 Q'A_3 \,.\, d_1 \\ &+ \kappa_1\kappa_2\kappa_3 Q'A_3 \,.\, d_1 \,.\, d_3. \quad \longleftarrow \end{aligned}$$

By addition

$$\begin{aligned} 1/m_Q + m_Q = 2 &+ (\kappa_1 + \kappa_2 + \kappa_3)\, QQ' \\ &+ \kappa_1\kappa_2 (QQ' - d_3)\, d_3 + \kappa_1\kappa_3 (QQ' - d_2)\, d_2 + \kappa_2\kappa_3 (QQ' - d_1)\, d_1 \\ &+ \kappa_1\kappa_2\kappa_3 (QQ' - d_2)\, d_3 d_1 \\ = 2 &- \kappa_1\kappa_2 d_3^2 - \kappa_1\kappa_3 d_2^2 - \kappa_2\kappa_3 d_1^2 - \kappa_1\kappa_2\kappa_3 d_1 d_2 d_3 + K \,.\, QQ'. \end{aligned}$$

If Q be H, the left-hand side of this equation is 2, and if Q be L, the left-hand side is -2. And therefore

$$\kappa_1\kappa_2 d_3^2 + \kappa_1\kappa_3 d_2^2 + \kappa_2\kappa_3 d_1^2 + \kappa_1\kappa_2\kappa_3 d_1 d_2 d_3 = 0, \text{ or } = 4,$$

according as H and H' are coincident, or as L and L' are coincident. In terms of the focal lengths these conditions are

$$f_1 d_1^2 + f_2 d_2^2 + f_3 d_3^2 + d_1 d_2 d_3 = 0 \text{ or } = 4 f_1 f_2 f_3.$$

(iii) *To find the power and the cardinal points of a system of n equal thin lenses arranged at equal intervals on the same axis.*

Let κ be the power of each lens, a the distance between consecutive lenses, u the distance of a point Q in front of the first lens.

We have the difference equation (§ 65)

$$(a_n - a_{n-1})/\kappa_n - (a_{n-1} - a_{n-2})/\kappa_{n-1} = y_n - y_{n-1} = a a_{n-1},$$

or

$$a_n - (2 + a\kappa)\, a_{n-1} + a_{n-2} = 0.$$

Put $2 + a\kappa = 2\cos\theta$, then the solution of this equation is of the form $a_n/a_0 = A\cos n\theta + B\sin n\theta$, where A and B are independent of n.

Put $n=0$, then $A=1$; put $n=1$, then $A\cos\theta + B\sin\theta = 1 + \kappa u$.

On substitution we obtain

$$a_n/a_0 = \{(1+\kappa u)\sin n\theta - \sin(n-1)\,\theta\}/\sin\theta.$$

The power of the system is therefore $\kappa\sin n\theta/\sin\theta$; and the positions of the cardinal points can be at once obtained.

When the lenses are divergent θ is imaginary, and we replace the trigonometrical by hyperbolic functions.

When the lenses are convergent and the numerical focal length is greater than $\frac{1}{4}a$, then θ is a real angle between 0 and π. As f approaches $\frac{1}{4}a$, θ approaches π, and evaluating the indeterminate form in this case we have

$$a_n/a_0 = (-)^{n-1}\{n(1+\kappa u) + (n-1)\}.$$

The first principal focus in this special case is $(2n-1)\,a/4n$ in front of the first lens; the second principal focus is obviously the same distance behind the last lens.

When the numerical focal length f is less than $\frac{1}{4}a$, then the solution is obtained by putting $2 - a/f = -2\cosh\theta$, or $a/f = 4\cosh^2\frac{1}{2}\theta$, and then

$$a_n/a_0 = (-)^{n-1}\left\{\left(1-\frac{u}{f}\right)\sinh n\theta + \sinh(n-1)\,\theta\right\}\Big/\sinh\theta.$$

72. Normal adjustment.

In many optical instruments consisting of lenses on the same axis, *e.g.* telescopes, the intervals between the lenses are often so arranged that K vanishes. The system of lenses is then said to be in *normal adjustment*, and is sometimes spoken of as telescopic. The formulæ (I) and (II), Art. 64, shew that the principal foci and the unit points are then at infinity, and the geometrical constructions given above (§ 54) no longer hold good.

Since K is zero, Cotes's formulæ give

$$l/l' = a_n/a_0 = \frac{\partial K}{\partial \kappa_1}, \quad l'/l = a_0/a_n = \frac{\partial K}{\partial \kappa_n}.$$

The linear magnification is therefore constant for all positions of the object.

The relation between the distance u of any point Q in front of the first lens and the distance v of its geometrical focus behind the last lens may still be derived from the equation

$$u\frac{\partial K}{\partial \kappa_n}+v\frac{\partial K}{\partial \kappa_1}+\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n}=0$$

given in Art. 67; or more directly by the formula for the apparent distance given in Art. 86.

We may also proceed as follows. It is always possible to divide the system into two parts of finite powers. The necessary and sufficient condition for normal adjustment is that the second principal focus of the first part coincide with the first principal focus of the second part. Then rays entering parallel to the axis will emerge parallel to the axis, and conversely; the principal foci of the entire system, in other words, are at infinity.

Let F_1, F_2 be the principal foci of the first part, F its (algebraic) focal length; let f_1, f_2 be those of the second part, f its (algebraic) focal length. Let Q be an origin of light on the axis, Q_1 its conjugate in the first part, Q' its conjugate in the entire system.

We have then $$F_1Q\,.\,F_2Q_1=-F^2,$$
$$f_1Q_1\,.\,f_2Q'=-f^2,$$
and therefore, since F_2 and f_1 coincide,
$$f_2Q'/F_1Q=(f/F)^2 \qquad\text{(i).}$$

Determine the point C which is its own conjugate. It divides the line F_1f_2 externally in the ratio $F^2:f^2$, and therefore always exists finitely and uniquely unless $F^2=f^2$.

Excluding this latter case, we may call C the *optical centre* of the system, and deduce from (i) the equation
$$CQ'/CQ=(f/F)^2 \qquad\text{(ii).}$$

Any two conjugate foci Q and Q' therefore always lie on the same side of C.

Again, the linear magnification is the product of the linear magnifications in the two parts of the system. These are respectively F_2Q_1/F_2H' and f_1h/f_1Q_1. Hence the linear magnification is given by
$$l'/l=f_1h/F_2H'=-f/F \qquad\text{(iii).}$$

If $P'Q'$ be the image of any small object PQ, then the ratio of the angle $P'CQ'$ to the angle PCQ is the *angular magnification of the object, when viewed from C.* This ratio

$$=\frac{P'Q'}{CQ'}\Big/\frac{PQ}{CQ}=-F/f \quad\text{......................(iv)},$$

from (ii) and (iii), and is therefore independent of the position of the object.

For reasons given below (Art. 97) this constant ratio $(-F/f)$ is called the *magnifying power* of the telescopic system. If we denote it by M, then from (I) and (II), Art. 64,

$$M=\frac{\partial K}{\partial \kappa_1}=\left(\frac{\partial K}{\partial \kappa_n}\right)^{-1}.$$

Finally, as an exceptional case, no point C exists if $F^2=f^2$.

We have then $F_1Q=f_2Q'$, and therefore the distance QQ' between any two conjugate foci is constant and equal to F_1f_2. The image of any object is equal to it, and erect if $F=-f$, but inverted if $F=+f$; or according as $\frac{\partial K}{\partial \kappa_1}$ and its reciprocal $\frac{\partial K}{\partial \kappa_n}$ are ± 1.

73. Example.

Shew that four thin lenses on the same axis of powers $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, separated by intervals a, b, c will produce an image of every object coincident with the object in all respects if the distances apart be given by the equations

$$a\{\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}-\kappa_1\kappa_2\}=b\{-\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}-\kappa_2\kappa_3\}=c\{\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}-\kappa_3\kappa_4\}$$
$$=\kappa_1+\kappa_2+\kappa_3+\kappa_4,$$

where the radical may have either sign.

Let the system be divided into the two parts composed respectively of the first pair of lenses and of the second pair.

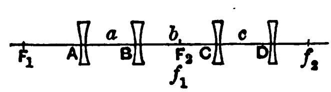


Fig. 46.

With the notation of the previous article it is necessary (i) that the entire system be in normal adjustment, (ii) that $F=-f$, (iii) that the interval F_1f_2 be zero.

Then if $F=-f$, $\quad \kappa_1+\kappa_2+a\kappa_1\kappa_2=-(\kappa_3+\kappa_4+c\kappa_3\kappa_4)=K$ say............(i).

The distance (v) of F_2 behind the second lens B is given by

$$0=1+\kappa_1 a+Kv;$$

the distance (u') of f_1 in front of the third lens C is given by

$$0=1+\kappa_4 c-Ku'.$$

Hence, subtracting, the condition that F_2 and f_1 coincide for normal adjustment gives

$$\kappa_1 a - \kappa_4 c + Kb = 0 \qquad \text{(ii)}.$$

Again the distance (u) of F_1 in front of A is given by

$$0 = 1 + \kappa_2 a + Ku;$$

and the distance (v') of f_2 behind D by $0 = 1 + \kappa_3 c - Kv'$.

If F_1, f_2 coincide, $u + v' + a + b + c = 0$.

Hence, subtracting, the third condition is

$$\kappa_2 a - \kappa_3 c - K(a + b + c) = 0 \qquad \text{(iii)}.$$

Adding (ii) and (iii), we obtain

$$a(\kappa_1 + \kappa_2 - K) = c(\kappa_3 + \kappa_4 + K),$$

whence

$$a^2 \kappa_1 \kappa_2 = c^2 \kappa_3 \kappa_4 \qquad \text{(iv)}.$$

But if s denote $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$, then (i) is

$$s + a\kappa_1\kappa_2 = -c\kappa_3\kappa_4 = \pm a\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4},$$

or

$$s/a = -\kappa_1\kappa_2 \pm \sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}.$$

Similarly $s/c = -\kappa_3\kappa_4 \pm \sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}$.

In these two equations the ambiguities have the same sign; also

$$c/a = \mp \sqrt{\kappa_1\kappa_2/\kappa_3\kappa_4}.$$

To determine b, we have from equations (i) and (ii)

$$a\kappa_1\kappa_2(\kappa_3 + \kappa_4) - c\kappa_3\kappa_4(\kappa_1 + \kappa_2) = K(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) = (\kappa_4 c - \kappa_1 a)\, s/b.$$

Hence

$$\begin{aligned} s/b &= -\{a\kappa_1\kappa_2(\kappa_3 + \kappa_4) - c\kappa_3\kappa_4(\kappa_1 + \kappa_2)\}/(a\kappa_1 - c\kappa_4) \\ &= -\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}\{\sqrt{\kappa_1\kappa_2}(\kappa_3 + \kappa_4) \pm \sqrt{\kappa_3\kappa_4}(\kappa_1 + \kappa_2)\}/\{\kappa_1\sqrt{\kappa_3\kappa_4} \pm \kappa_4\sqrt{\kappa_1\kappa_2}\} \\ &= -\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}(\sqrt{\kappa_2\kappa_3} \pm \sqrt{\kappa_1\kappa_4})/\sqrt{\kappa_1\kappa_4} \\ &= -\kappa_2\kappa_3 \mp \sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}. \end{aligned}$$

74. Combination of Systems.

To obtain the power and the cardinal points of a combination of given lens-systems on the same axis.

First, if all the systems have finite powers, then either of the methods given in Arts. 68 and 69 for a combination of thick lenses is directly applicable.

In the case of only two systems it is instructive to determine the cardinal points by direct use of the geometrical constructions.

Secondly, if one of the systems be telescopic, we can combine a telescopic system with one of finite power as follows.

Let the finite system precede the telescopic; let its cardinal points be f_1, f_2, h, h' and its power K. Let C be the optical centre

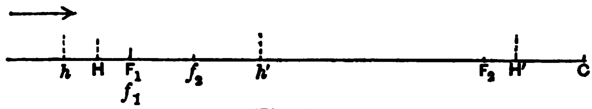


Fig. 47.

of the telescope, M its magnifying power. Denote as usual the cardinal points of the combination by F_1, F_2, H and H'.

Since rays from f_1 will emerge parallel to the axis from the first system, and will retain this property after passing through the telescope, F_1 coincides with f_1.

The second focus F_2 will be the image of f_2 in the telescope, *i.e.* F_2 is given by $CF_2 = Cf_2/M^2$.

Again the linear magnification for the combination is given by

$$l'/l = (f_1h/f_1Q)(1/M) \equiv F_1H/F_1Q.$$

Hence the unit point H is given by $F_1H = f_1h/M$, and therefore

$$F_2H' = f_2h'/M.$$

The power of the combination is therefore $K \,.\, M$.

When the telescopic precedes the finite system, it will be sufficient to exchange the words first and second in the sentences above. The power of the combination will however be K/M, since the magnifying power of a telescope is inverted if the light traverse it in the reversed direction.

Thirdly, if all the systems be telescopic, the resulting system is telescopic.

The magnifying power of the resulting system is obviously the product of the magnifying powers of the given systems.

Let $c_1, c_2 \dots c_{n-1}$ be the algebraic distances apart of their centres $C_1, C_2 \dots C_n$ taken in order. Then if u be the distance of an origin of light Q in front of C_1, $u_1, u_2 \dots u_n$ the distances of its successive geometrical foci in front of $C_1, C_2, C_3 \dots C_n$ respectively, we have

$$u_1 = u/M_1^2;\ u_2 = (u_1 + c_1)/M_2^2;\ \dots\ u_n = (u_{n-1} + c_{n-1})/M_n^2.$$

Hence, $u_n = c_{n-1}/M_n^2 + c_{n-2}/M_n^2 M_{n-1}^2 + \dots + u/M_1^2 M_2^2 \dots M_n^2$.

By putting $u_n = c_1 + c_2 + \dots + c_{n-1} + u$, the position of the centre of the combination is at once determined from this equation.

EXAMPLES.

1. Prove the following construction (due to Möbius) for the image of a given object in any optical instrument in air.

At the principal foci F_1, F_2 draw F_1G_1, F_2G_2 perpendicular to the axis and each equal to the focal length; on G_1G_2 as diameter describe a circle, and from any point O on this circle draw OG_1, OG_2 cutting the axis in Q and Q', and draw OCD, cutting the axis in C, to D, the end of the diameter of the circle perpendicular to the axis, nearer or further from O, as the image is erect or inverted. Then Q, Q' are conjugate foci, and the lines joining corresponding points of the object and image all pass through C.

2. Shew that for two thin lenses of focal lengths f_1, f_2 on the same axis at distance a apart, there are two points such that each is its own image if $a^2 > 4f_1f_2$.

Shew that, if $a^2 = 4f_1f_2$, these points coincide, and, if O be the point, the relation between any two conjugate foci is

$$1/OQ - 1/OQ' = 1/f_1 + 1/f_2 + a/f_1f_2;$$

and the linear magnification is $-OQ'/OQ$.

3. Shew that any coaxial system of lenses can be completely replaced by two thin lenses.

If the focal length of the system be f and the distance between its unit points be c, the focal lengths of the two lenses, supposed at distance a apart, are the roots of the equation

$$cx^2 - a(a-c)x + a^2f = 0.$$

4. Shew that a thick lens of index μ and thickness $\mu\tau$ may be replaced in all respects by two thin lenses at a distance a apart, if their powers be the roots of the equation

$$a^2z^2 - \{(\kappa_1+\kappa_2)(a-\overline{\mu-1}\,\tau)+\kappa_1\kappa_2\tau(a-\mu\tau)\}az+\{(\mu-1)(\kappa_1+\kappa_2)+\kappa_1\kappa_2\mu\tau\}\tau=0,$$

where κ_1, κ_2 are the powers of the two surfaces of the lens.

5. In an optical instrument of convergent focal length f, the distance between the unit points is $4h$. Prove that it can be made equivalent in all respects to a single thin lens by placing a thin divergent lens of focal length $f+h$ midway between the unit points, provided this position fall behind the instrument.

6. A photographic camera has a single lens of convergent focal length F. It is desired to enlarge n times the linear dimensions of the picture of any object. Shew that this can be done by placing in front of the lens a doublet of convergent focal length f, provided that the distance between the unit points of the doublet be $(n-1)^2 f/n$, and that the second focus of the doublet fall at a distance f/n behind the first focus of the lens, while the distance of the plate from the lens must be $F - nF^2/f + n^2F^2/(u-F)$, where u is the distance of the object from the camera.

7. The distance between the image of any object formed by a system of thin lenses and the image formed by their equivalent lens is

$$f\Sigma\kappa_r\kappa_s\ldots\kappa_w\,.\,a_{rs}\,.\,a_{st}\ldots a_{vw}a_{rw},$$

where $r<s<\ldots<w$, a_{rs} is the distance between the lenses of powers κ_r and κ_s, f is the equivalent focal length, and the summation extends to all possible combinations of the powers.

8. A system of n convergent thin lenses are arranged on the same axis so that the distance between any consecutive pair is twice the sum of their numerical focal lengths; then the power of the system is

$$(-)^{n-1}(\kappa_1+\kappa_2+\ldots+\kappa_n).$$

9. Prove that if an even number $2n$ of thin convergent lenses be arranged on the same axis so that each alternate interval, counting from the first, is the sum of the numerical focal lengths of the lenses terminating that interval, the linear magnification is constant and equal to

$$(-)^n f_2f_4\ldots f_{2n}/f_1f_3\ldots f_{2n-1}.$$

If an additional convergent lens of numerical focal length f_{2n+1} be added the focal length of the entire system is

$$(-)^{n+1}f_1f_3\ldots f_{2n+1}/f_2f_4\ldots f_{2n}.$$

10. If n thin convergent lenses of numerical focal length f be arranged on the same axis at a constant distance $3f$ apart, then the focal length of the system is $\pm f$ as n is of the forms $3m-1$ or $3m+1$; while if $n=3m$ the linear magnification is always $+1$, and the distance between any object and its image is $3nf$.

11. A system of $2n$ thin convergent lenses of numerical focal length f are placed on the same axis at distance $4f$ apart except the two middle ones which are at a distance $8f$ apart. Shew that the focal length of a lens which must be placed midway between the two middle ones so that the image of a bright point at a distance $4f$ in front of the first lens may be formed at an equal distance behind the last lens is $2(n+1)f/(2n+1)$.

12. If n thin lenses each of focal length f be placed on the same axis at regular intervals a, where a is small compared with f, then the distances of any two conjugate foci from the middle point of the system are connected by the approximate equation

$$\frac{1}{v}-\frac{1}{u}=\frac{n}{f}+\frac{n(n^2-1)}{6}\frac{a}{f^2},$$

where squares &c. of a/f have been neglected.

13. Given the cardinal points and planes of two thick lenses, or of two systems of lenses, on the same axis, determine geometrically the cardinal points of the combination.

Prove that, if F_1, F_2, and F_1', F_2' be the principal foci of the two systems, f, f' their focal lengths, the combination may be replaced by a single thin lens provided
$$(f+f')^2+F_1F_2' . F_2F_1'=0.$$

14. Two optical instruments, each symmetrical about an axis, are placed with their axes in a straight line, and the power of the combination is K. If both instruments are inverted (the positions on the axis of the extreme lenses in each instrument being unaltered), the power is K', and if one instrument only is inverted, the power is K_1 or K_1', according to the one inverted.

Shew that
$$K+K'=K_1+K_1'.$$

15. A thick double convex lens is placed in contact with another thick lens. The combination is of focal length f or f' as one or other of the spherical faces of the first lens touches the same face of the second.

Shew that the focal length of the second lens is

$$\left(1-\frac{1}{\mu}\right)t\left(\frac{1}{s}-\frac{1}{r}\right)\Big/\left(\frac{1}{f}-\frac{1}{f'}\right),$$

where t is the thickness and r and s the radii of the surfaces of the first lens.

16. Shew that if three thin lenses can be arranged so that the image of every object coincides in position with the object, they must be convergent, and, if d_3, d_1 and d_2 ($\equiv d_3+d_1$) be their distances apart, their focal lengths are $-d_2d_3/2d_1$, $-d_3d_1/2d_2$ and $-d_1d_2/2d_3$ respectively.

17. Three thin lenses of focal lengths f_1, f_2, f_3 on the same axis are placed at such intervals d_3, d_1 that the image of every object is at a constant distance from the object: prove that either

$$\text{(i)}\quad f_1d_1=f_3d_3=-(f_1f_2+f_2f_3+f_3f_1),$$

and then the constant distance is $-(f_1f_2+f_2f_3+f_3f_1)^2/f_1f_2f_3$,

or
$$\text{(ii)}\quad f_1d_1-2f_2f_3=f_3d_3-2f_1f_2=-(f_1f_2+f_2f_3+f_3f_1),$$

and then the constant distance is

$$\frac{f_2f_3}{f_1}+\frac{f_3f_1}{f_2}+\frac{f_1f_2}{f_3}-2f_1-2f_2-2f_3.$$

18. Three coaxial systems of focal lengths f_1, f_2, f_3 form a system in normal adjustment. Shew that if c_1, c_2 be the distances between the second focus of one system and the first focus of the next system, $f_2^2=c_1c_2$; and if d be the distance between the first focus of the first system and the second focus of the last, the optical centre of the entire system divides the distance d externally in the ratio

$$f_1^2f_3^2+f_1^2c_2d : f_1^2f_3^2+f_3^2c_1d.$$

19. If four thin lenses of powers κ_1, κ_2, κ_3, κ_4 on the same axis be so arranged that they give the image of every object coincident with the object in position but inverted, the intervals a, b, c between them in order are given by the equations

$$a(\kappa_1\kappa_2+\kappa_3\kappa_4)=\kappa_3+\kappa_4-\kappa_1-\kappa_2+R/\kappa_1\kappa_2,$$

$$c(\kappa_1\kappa_2+\kappa_3\kappa_4)=\kappa_1+\kappa_2-\kappa_3-\kappa_4+R/\kappa_3\kappa_4,$$

$$b(\kappa_1\kappa_4+\kappa_2\kappa_3)=\kappa_1-\kappa_2-\kappa_3+\kappa_4-R/\kappa_2\kappa_3,$$

where $$R^2=\{4(\kappa_1\kappa_2+\kappa_3\kappa_4)-(\kappa_1+\kappa_2-\kappa_3-\kappa_4)^2\}\kappa_1\kappa_2\kappa_3\kappa_4,$$

and the sign to be given to R is that of $\kappa_1\kappa_2\kappa_3\kappa_4$.

20. The intervals between n thin coaxial lenses, of focal lengths f_1, $f_2 \ldots f_n$, are a_1, $a_2 \ldots a_{n-1}$ in order. Shew that the power of the system is the determinant of $(n-1)$ rows and columns

$$\frac{1}{f_1f_2\ldots f_n}\cdot\begin{vmatrix} f_1+f_2+a_1, & f_2, & 0\ldots & & 0 & 0 \\ f_2, & f_2+f_3+a_2, & f_3, & & 0 & 0 \\ 0 & \ldots\ldots\ldots\ldots & & & & \\ & & & f_{n-2}, & f_{n-2}+f_{n-1}+a_{n-2}, & f_{n-1} \\ 0\ldots & 0\ldots\ldots\ldots & & & f_{n-1}, & f_{n-1}+f_n+a_{n-1} \end{vmatrix}.$$

21. Any two rays, making small angles with the axis, pass through a refracting system of finite focal length placed in air. If b be the distance between the points in which the incident rays cut the first focal plane, b' the distance between the points in which the emergent rays cut the second focal plane, d the shortest distance between the incident rays and d' that between the emergent rays, shew that

$$d/b=d'/b'.$$

22. Shew that if two star photographs be taken by two telescopes so nearly parallel that some stars are found on both plates, and if (x, y), (x', y') be the two coordinates of the same star referred to any system of axes on the two plates, the relations between these coordinates are of the form

$$x'=ax+by+c, \qquad y'=bx-ay+d,$$

where a, b, c, d are constants which do not depend on any particular star.

COAXIAL SPHERICAL SURFACES.

75. Let there be n coaxial spherical surfaces, vertices $A_1, A_2 \dots A_n$, separating media of refractive indices $\mu_0, \mu_1 \dots \mu_n$; and let their radii be $\rho_1, \rho_2 \dots \rho_n$, reckoned positive when the surfaces are concave towards the side on which the light is incident.

Their powers are given by the equations

$$\kappa_1 = (\mu_1 - \mu_0)/\rho_1, \quad \kappa_2 = (\mu_2 - \mu_1)/\rho_2 \dots \kappa_n = (\mu_n - \mu_{n-1})/\rho_n.$$

Let the interval between any two consecutive surfaces be divided by the index of the medium in which it lies; this is

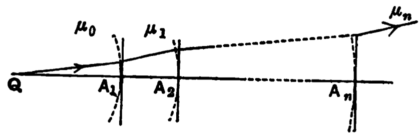


Fig. 48.

called the *equivalent thickness*, and for example, A_1A_2/μ_1 is written $\overline{A_1A_2}$.

Also let $\overline{QA_1}$ denote QA_1/μ_0, $\overline{Q'A_n}$ denote $Q'A_n/\mu_n$.

If a ray from a point Q on the axis cross the surfaces at heights $y_1, y_2 \dots y_n$, and have successive angles of divergence $\alpha_0, \alpha_1, \alpha_2 \dots \alpha_n$, we can form the equations

$$\left.\begin{array}{r} y_1 = QA_1 \,.\, \alpha_0 = \overline{QA_1} \,.\, \mu_0\alpha_0 \\ \mu_1\alpha_1 - \mu_0\alpha_0 = \kappa_1 y_1 \\ y_2 - y_1 = A_1A_2 \,.\, \alpha_1 = \overline{A_1A_2} \,.\, \mu_1\alpha_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \mu_n\alpha_n - \mu_{n-1}\,\alpha_{n-1} = \kappa_n y_n \end{array}\right\} \dots\dots\dots\dots \text{(i).}$$

These equations are of exactly the same type as those in Art. 64, and we deduce the formulæ (cf. Arts. 64, 65, 66),

$$\mu_n\alpha_n/\mu_0\alpha_0 = K \,.\, \overline{QA_1} + \frac{\partial K}{\partial \kappa_1},$$

$$= 1 + \Sigma\kappa_r\kappa_s \dots \overline{QA_r} \cdot \overline{A_rA_s} \dots \longrightarrow \text{(I)},$$

where $\overline{QA_r}$ denotes

$$QA_1/\mu_0 + A_1A_2/\mu_1 + A_2A_3/\mu_2 + \dots + A_{r-1}A_r/\mu_{r-1},$$

and $\overline{A_rA_s}$ denotes $A_rA_{r+1}/\mu_r + A_{r+1}A_{r+2}/\mu_{r+1} + \dots + A_{s-1}A_s/\mu_{s-1}$.

Similarly by retracing the path of a ray,

$$\mu_0\alpha_0/\mu_n\alpha_n = K.\overline{Q'A_n} + \frac{\partial K}{\partial\kappa_n},$$

$$= 1 + \Sigma\kappa_s\kappa_r \ldots \overline{Q'A_s}.\overline{A_sA_r} \ldots \longleftarrow \quad \text{(II)}.$$

Also $$K = \Sigma\kappa_r + \Sigma\kappa_r\kappa_s\kappa_t \ldots \overline{A_rA_s}.\overline{A_sA_t} \ldots \quad \text{(III)}.$$

This quantity, K, as before, is defined as the *power* of the system. We shall assume that it is not zero.

The formulæ (I, II) of Art. 49 given for the thick lens, are the first and simplest examples of the above theorems.

76. Focal Lengths and Cardinal Points.

Since by Helmholtz's formula $\mu_n l'\alpha_n = \mu_0 l\alpha_0$, we see that as before (I) is a formula for the reciprocal of the linear magnification, and (II) for the linear magnification.

We may use (I) to determine the position of the first unit point H by putting $l = l'$, and of the first principal focus F_1 by putting α_n equal to zero.

Since $$\mu_n\alpha_n/\mu_0\alpha_0 = l/l' = K.\overline{QA_1} + \frac{\partial K}{\partial\kappa_1}, \longrightarrow$$

$$1 = K.\overline{HA_1} + \frac{\partial K}{\partial\kappa_1},$$

$$0 = K.\overline{F_1A_1} + \frac{\partial K}{\partial\kappa_1},$$

we deduce $$1 = K.\overline{HF_1} \longrightarrow \quad \ldots\ldots\text{(i)},$$

and $$l/l' = QF_1/HF_1 \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Similarly from the equations

$$\mu_0\alpha_0/\mu_n\alpha_n = l'/l = K.\overline{Q'A_n} + \frac{\partial K}{\partial\kappa_n}, \longleftarrow$$

$$1 = K.\overline{H'A_n} + \frac{\partial K}{\partial\kappa_n},$$

$$0 = K.\overline{F_2A_n} + \frac{\partial K}{\partial\kappa_n},$$

we deduce $$1 = K.\overline{H'F_2} \longleftarrow \quad \ldots\ldots\text{(iii)},$$

and $$l'/l = Q'F_2/H'F_2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

In equations (i) and (iii) the directions of measurement are opposed, and the first and last media in which the equivalent distances are taken are not necessarily the same. Hence the actual distances HF_1 and F_2H' are no longer equal, but are to each other in the ratio $\mu_0 : \mu_n$, *i.e.* in the ratio of the refractive indices of the initial and final media. These distances are sometimes called the *focal lengths*, and denoted by f_1 and f_2.

From (ii) and (iv) we deduce the equation

$$F_1Q \,.\, F_2Q' = F_1H \,.\, F_2H' = -f_1 \,.\, f_2 = -\mu_0\mu_n/K^2 \ldots\ldots \text{(IV)}.$$

Also from the first two equations above, we obtain, on subtraction, the equation

$$\mu_n\alpha_n - \mu_0\alpha_0 = Ky \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(V)},$$

where y is the height above the axis of the points where the incident and the emergent ray meets the unit planes.

Lastly, we may determine the positions of the *nodal points* N and N', which are defined by the property that any incident ray crossing the axis at N crosses the axis at N' in the same direction after emergence. Hence for N and N' we have $\alpha_n = \alpha_0$, $l'/l = \mu_0/\mu_n$.

Putting $\alpha_n = \alpha_0$ in (I), we have

$$\mu_n/\mu_0 = K \,.\, \overline{NA_1} + \frac{\partial K}{\partial \kappa_1} \quad \longrightarrow$$

$$= K \,.\, \overline{NF_1} = NF_1/HF_1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)};$$

and similarly

$$\mu_0/\mu_n = K \,.\, \overline{N'A_n} + \frac{\partial K}{\partial \kappa_n} \quad \longleftarrow$$

$$= K \,.\, \overline{N'F_2} = N'F_2/H'F_2 \ldots\ldots\ldots\ldots \text{(vi)}.$$

Hence $$NF_1 = F_2H' \text{ and } HF_1 = F_2N' \ldots\ldots\ldots\ldots \text{(vii)}.$$

These relations connecting the distances between the cardinal points and the formulæ for the linear magnification are easily proved by the geometrical constructions of the following article.

77. Geometrical Constructions.

First, assuming any positions on the axis for F_1, H and H', let N be the first nodal point.

Draw two incident parallel rays through F_1 and N respectively, meeting the first unit plane in R and T. The emergent ray

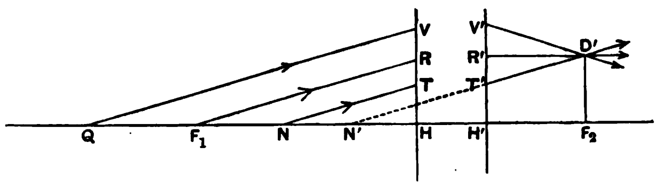


Fig. 49.

corresponding to the ray NT is unaltered in direction, and meets the second unit plane in T', where $H'T' = HT$.

Hence $$NN' = TT' = HH' \quad\text{.....................} \quad \text{(i)}.$$

The emergent ray corresponding to F_1R is parallel to the axis, and passes through R', where $H'R' = HR$; the two emergent rays through T' and R' must meet in a point D' on the second focal plane, since the incident rays were parallel. It is obvious from the equal triangles F_1HR and $N'F_2D'$ that

$$F_1H = N'F_2 \quad\text{........................} \quad \text{(ii)}.$$

Hence also $$F_1N = H'F_2.$$

Again, let a third ray in the same direction before incidence meet the axis in H, then the corresponding emergent ray is $H'D'$ and by the definition of H and H', we have, for this ray, $\mu_0\alpha_0 = \mu_n\alpha_n$;

i.e. $$\mu_0 (HF_1R) = \mu_n (F_2H'D')$$

and therefore $$\mu_0 HR/HF_1 = \mu_n F_2D'/F_2H'.$$

Hence $$HF_1/\mu_0 = F_2H'/\mu_n \quad\text{.....................} \quad \text{(iii)}.$$

Again, to determine the emergent ray corresponding to any incident ray QV, which is parallel to F_1R or NT, and meets the first unit plane in V, the emergent ray meets the second unit plane in V' and passes through the point D' on the second focal plane, already determined (cf. Fig. 49).

Lastly, to determine the image $P'Q'$ of the small object PQ perpendicular to the axis, we may make use of the rays PR through F_1, and PS parallel to the axis, emerging respectively in the directions $P'R'$ parallel to the axis, and $P'S'$ through F_2. We

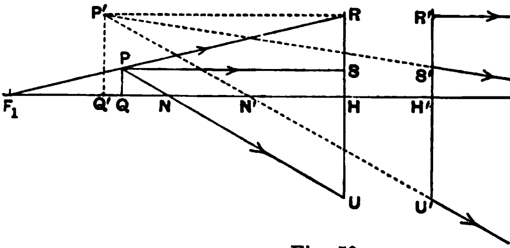


Fig. 50.

may also draw the ray PNU, then the emergent ray $P'N'U'$ is parallel to it (cf. Fig. 50).

From similar triangles it is obvious that

$$\frac{P'Q'}{PQ} = \frac{F_2Q'}{F_2H'} = \frac{F_1H}{F_1Q} = \frac{N'Q'}{NQ} = \frac{H'Q'/\mu_n}{HQ/\mu_0} \quad \text{.........} \quad \text{(iv)};$$

where the last equation follows from the fact that if the angle $PHQ = \alpha_0$, the angle $P'H'Q'$ is the corresponding value α_n, and that $\mu_0\alpha_0 = \mu_n\alpha_n$ for the points H and H'.

Figures (49) and (50) have been drawn for a system of refracting media in which μ_n has the same sign as μ_0 (*i.e.* when among the refractions no reflections or an even number only are included, cf. Art. 83), and in which K is negative. The formulæ hold good in any case.

78. Equivalent Spherical Surface.

It is obvious from (V) Art. 76 that a spherical surface, having its vertex at H and separating the first and last media μ_0, μ_n, will produce the same deviation as the system in any ray if its power be K; *i.e.* if its curvature be $K/(\mu_n - \mu_0)$. The centre of this surface will be at N, its first focus at F_1, and the images of any object produced by the system and by it are exactly equal, but separated by the distance HH'.

79. Formula for the elongation.

Let Q and R be two points on the axis, Q' and R' their geometrical foci; the ratio $Q'R'/QR$ is the elongation (cf. Art. 55).

Since $$F_1Q\,.\,F_2Q' = F_1R\,.\,F_2R' = F_1H\,.\,F_2H',$$

it follows that $$Q'R' = (F_1H\,.\,F_2H')\left(\frac{1}{F_1R} - \frac{1}{F_1Q}\right)$$

$$= -\frac{F_1H\,.\,F_2H'}{F_1Q\,.\,F_1R}\,.\,QR.$$

Hence $$\frac{Q'R'}{QR} = \frac{F_2H'}{HF_1}\,.\,\frac{(F_1H)^2}{F_1Q\,.\,F_1R} = \frac{\mu_n}{\mu_0} m_Q m_R \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\text{(i)},$$

where m_Q, m_R are the linear magnifications for Q and R respectively.

We can also deduce a formula connecting the distances of Q and Q' from R and R' as follows.

Since $$(F_1R + RQ)(F_2R' + R'Q') = F_1R\,.\,F_2R',$$

it follows that $$F_1R/RQ + F_2R'/R'Q' + 1 = 0,$$

whence $$\mu_n m_R/R'Q' - \mu_0 m_R^{-1}/RQ + K = 0 \;\ldots\ldots\;\text{(ii)}.$$

If we put R and R' at H and H', this reads

$$\mu_n/H'Q' - \mu_0/HQ = \mu_n/H'F_2 = \mu_0/F_1H \;\ldots\ldots\ldots\ldots\ldots\ldots\;\text{(iii)},$$

and if we put R and R' at N and N', then this reads

$$\mu_0/N'Q' - \mu_n/NQ = \mu_0/N'F_2 = \mu_n/F_1N \;\ldots\ldots\ldots\ldots\ldots\ldots\;\text{(iv)};$$

the terms involving F_1, F_2 having been obtained by taking Q and Q' successively at infinity.

80. Paths of rays which do not meet the axis.

Let μ and μ' be used for the indices of refraction of the initial and final media; and let an incident ray be defined, as in Art. 56, by its direction cosines $(l, m, 1)$ and the coordinates (ξ, η) of the point in which it meets the first unit plane. Then the emergent ray is defined by its direction-cosines $(l', m', 1)$, and by the coordinates, also (ξ, η), of the point in which it meets the second unit plane. The constructions given in Art. 56 for the emergent rays hold good in all points, but the equations obtained by projection are

$$\xi - \xi_0 = lF_1H = -l\mu/K;\quad \eta - \eta_0 = -m\mu/K,$$

and $$-\xi_0 = l'H'F_2 = -l'\mu'/K;\quad -\eta_0 = -m'\mu'/K.$$

Hence $$\left.\begin{aligned}\mu'l' - \mu l &= K\xi\\ \mu'm' - \mu m &= K\eta\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\text{(i)}.$$

To find the coordinates (x', y') of the geometrical focus P' of a point P, of coordinates (x, y), let z be the distance from P to the first unit plane, and z' the distance from P' to the second unit plane, both reckoned positive when taken onwards with the light, then we have

$$\xi = x + lz = x' + l'z',$$

$$\eta = y + mz = y' + m'z'.$$

Substitute in the first equation for l' and ξ; then

$$x + lz = x' + z'\{\mu l + K(x + lz)\}/\mu'.$$

Since P' is the geometrical focus of P, this equation must be true, to our order of approximation, for all rays. Hence equating the coefficients of l, we have $\mu'z - \mu z' = Kzz'$ (cf. iii § 79), and from the other terms, and the similar equations in y, we deduce

$$\frac{x'}{x} = \frac{y'}{y} = \frac{f_2 - z'}{f_2} = \frac{f_1}{f_1 + z} = \frac{z'/\mu'}{z/\mu} \quad \text{(ii)},$$

where f_1, f_2 have been written for μ/K and μ'/K respectively.

These equations also follow, as before, from the fact that P and P' lie in the same plane through the axis, and from the geometrical constructions.

81. Example.

The distances of the principal foci of a coaxial system of lenses, placed in air, from the first and last lenses are u_1 and u_2, estimated positive when measured away from the system, and f is the focal length of the system. Media of refractive indices μ_1 and μ_2 are introduced before the first lens and after the last lens respectively; shew that the focal lengths of the system are f_1 and f_2, where

$$\frac{\mu_1}{f_1} = \frac{\mu_2}{f_2} = \frac{1}{f}\left\{1 + \frac{\mu_1 - 1}{r_1} u_1 + \frac{\mu_2 - 1}{r_2} u_2 + \frac{(\mu_1 - 1)(\mu_2 - 1)}{r_1 r_2}(u_1 u_2 - f^2)\right\},$$

and the positions of the principal foci of the system are given by u_1', u_2', where

$$\frac{(fu_1' - f_1 u_1) r_2}{(\mu_2 - 1) f_1} = \frac{(fu_2' - f_2 u_2) r_1}{(\mu_1 - 1) f_2} = u_1 u_2 - f^2,$$

and r_1, r_2 are the radii of the outer surfaces of the first and last lenses respectively, being positive when those surfaces are concave.

Let κ_1 and κ_2 be the powers of the first and last surfaces when the system is in air; κ_1', κ_2' their powers when the media are introduced. Then, if μ be the index of refraction of the material of the lenses, we have with the notation of the question, $\kappa_1 = (\mu - 1)/r_1$, $\kappa_1' = (\mu - \mu_1)/r_1$, and therefore $\kappa_1 - \kappa_1' = (\mu_1 - 1)/r_1$; and also $\kappa_2 = (1 - \mu)/(-r_2)$, $\kappa_2' = (\mu_2 - \mu)/(-r_2)$, $\kappa_2 - \kappa_2' = (\mu_2 - 1)/r_2$.

Again, let K be the power of the system when placed in air, and K' when the media are introduced. Then since K contains the power of each surface linearly, we may write it in the form $K = A + B\kappa_1 + C\kappa_2 + D\kappa_1\kappa_2$, where A, B,

C, D are independent of κ_1 and κ_2. The positions of the principal foci are given by the equations

$$0=Ku_1+\frac{\partial K}{\partial \kappa_1}=\frac{u_1}{f}+B+D\kappa_2,$$

$$0=Ku_2+\frac{\partial K}{\partial \kappa_2}=\frac{u_2}{f}+C+D\kappa_1.$$

Hence $$u_1u_2-f^2=f^2\left(\frac{\partial K}{\partial \kappa_1}\frac{\partial K}{\partial \kappa_2}-1\right)\equiv f\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_2}=fD \text{ (cf. § 67).}$$

In the second case we have $K'=\mu_1/f_1=\mu_2/f_2$, and the positions of the principal foci are now given by

$$0=K'u_1'/\mu_1+\frac{\partial K'}{\partial \kappa_1'}=\frac{u_1'}{f_1}+B+D\kappa_2',$$

and $$0=K'u_2'/\mu_2+\frac{\partial K'}{\partial \kappa_2'}=\frac{u_2'}{f_2}+C+D\kappa_1'.$$

Subtracting the previous equations,

$$\frac{u_1'}{f_1}-\frac{u_1}{f}=D(\kappa_2-\kappa_2')=\frac{u_1u_2-f^2}{f}\frac{\mu_2-1}{r_2}.$$

Similarly $$\frac{u_2'}{f_2}-\frac{u_2}{f}=D(\kappa_1-\kappa_1')=\frac{u_1u_2-f^2}{f}\frac{\mu_1-1}{r_1}.$$

Also $$K'-K=\frac{\mu_1}{f_1}-\frac{1}{f}\equiv\left(\frac{\partial K'}{\partial \kappa_1'}\frac{\partial K'}{\partial \kappa_2'}-\frac{\partial K}{\partial \kappa_1}\frac{\partial K}{\partial \kappa_2}\right)\bigg/ D$$

$$=\left(\frac{u_1'u_2'}{f_1f_2}-\frac{u_1u_2}{f^2}\right)\bigg/D$$

$$=\left\{\frac{\mu_1-1}{r_1}\frac{u_1}{f}+\frac{\mu_2-1}{r_2}\frac{u_2}{f}+\frac{(\mu_1-1)(\mu_2-1)}{r_1r_2}\frac{u_1u_2-f^2}{f}\right\}$$

on substituting from the equations above for u_1', u_2' and D.

82. Example.

The surfaces of equal density of a heterogeneous refracting medium are equal spheres, whose centres lie on the axis of z. The index of refraction at any point of the axis is $e^{2\kappa z}$, and the medium extends from the origin to a distance a in the positive direction along the axis of z, beyond which the medium is uniform and of index $e^{2\kappa a}$. If a small pencil of rays be directly refracted through this system, determine the cardinal points and focal lengths of the system.

The equations (i) of Art. 75 may be written, when small pencils are refracted directly through continuously varying media, as $d(\mu a)=d\mu\,.\,y/\rho$, $dy=adz$.

Hence we have for any law of refractive index

$$\frac{d}{dz}\left(\mu\frac{dy}{dz}\right)=\frac{d\mu}{dz}\frac{y}{\rho}.$$

If $\mu = e^{2\kappa z}$, this equation is

$$\frac{d^2y}{dz^2} + 2\kappa\frac{dy}{dz} - \frac{2\kappa y}{\rho} = 0.$$

Let the surfaces be convex to the light as it moves positively along the axis of z, and let $1/\rho = -(\kappa^2+m^2)/2\kappa$; then the solution of this differential equation is

$$y = e^{-\kappa z}(A\cos mz + B\sin mz).$$

If we determine the constants A and B by the values of y and $\frac{dy}{dz}$ (or a_0) at the origin, we have on putting $\kappa = m\tan\beta$, $\left(\rho = -\frac{1}{m}\sin 2\beta\right)$,

$$y = e^{-\kappa z}\left\{y_0\sec\beta\cos(mz-\beta) + \frac{a_0}{m}\sin mz\right\},$$

$$\mu a = e^{+\kappa z}\{-my_0\sec^2\beta\sin mz + a_0\sec\beta\cos(mz+\beta)\}.$$

Since the index of refraction of the final continuous medium is $e^{2\kappa a}$, there is no discontinuous refraction at the last surface, and these values of y and a hold for the emergent pencil. Putting $z=a$, and $y_0 = a_0 u$, we see that the power of the system, *i.e.* the coefficient of u in $\mu a/a_0$, is $-me^{\kappa a}\sec^2\beta\sin ma$; and the focal lengths are therefore given by

$$f_1 = -\frac{1}{m}e^{-\kappa a}\cos^2\beta\,\text{cosec}\,ma,$$

and

$$f_2 = -\frac{1}{m}e^{\kappa a}\cos^2\beta\,\text{cosec}\,ma.$$

The distance of the first principal focus F_1 from the origin is given by putting $a=0$; that is, F_1 is at a distance $\frac{1}{m}\cos\beta\cos(ma+\beta)\,\text{cosec}\,ma$ on the negative side of the origin; the distance of the second principal focus F_2 from the last refracting surface is the value of y/a, when a_0 is zero; *i.e.* F_2 is at a distance $\frac{1}{m}\cos\beta\cos(ma-\beta)\,\text{cosec}\,ma$ beyond the last surface.

The cardinal points of the system are therefore completely determined, and we have, if Q, Q' be any two conjugate foci,

$$F_1Q\,.\,F_2Q' = -\frac{1}{m^2}\cos^4\beta\,\text{cosec}^2\,ma.$$

We see that, if the thickness of the system be such that $ma = n\pi$, where n is an integer, it is in normal adjustment (cf. Art. 72). We have then

$$y = (-)^n e^{-\kappa a}y_0, \qquad \mu a = (-)^n e^{\kappa a}a_0.$$

The linear magnification $= a_0/\mu a = (-)^n e^{-\kappa a} = (-)^n\sqrt{1/\mu}$; the distance of the image from the last surface is y/a, which is equal to y_0/a_0; the distance therefore between any two conjugate foci is constant and equal to a.

83. Reflection at Spherical Surfaces.

The laws of reflection may be considered as a particular case of those of refraction, obtained by putting $\mu' = -\mu$ in the latter. The quasi-refracted ray must however be taken as moving in the direction beyond the reflecting surface, which is opposite to that of the true reflected ray. If this be done, the formulæ of Arts. 4 and 19 giving the direction-cosines of the ray will be found to agree; and the angle of divergence of a ray after reflection at a spherical surface will be given by the standard formula of Art. 46, with all the conventions of that article. The power, for instance, of a convex spherical mirror in air will be $(-1-1)/(-\rho)$, and its focal length $\frac{1}{2}\rho$; so that the definition of focal length of a mirror given in Art. 36 agrees with that used in this chapter for refracting surfaces. Hence, if a pencil in passing through refractive media be directly reflected at a spherical surface we can obtain its final focus by applying the preceding formulæ, provided that we treat all the indices of refraction that occur after the reflection as negative, until another reflection occurs.

Although the direction of the light is really reversed by the reflection, it must be considered as travelling onwards in the general direction of incidence, and its actual path treated as if virtual, while the succeeding surfaces must be considered to lie behind each other on the axis.

These details will be made evident by the following example.

84. The back of a lens of thickness t is silvered; to find the power of the equivalent spherical mirror and its position.

If a pencil after traversing any number of refractive media separated by coaxial spherical surfaces be reflected at a spherical surface on the same axis, and retraverse all the media in opposite order, the entire system is in all respects equivalent to a spherical mirror.

A pencil of rays incident parallel to the axis will finally converge to F_2 on the axis. If we retrace the paths of these rays from F_2 all the refractions and the reflection take place in their proper order, and the rays emerge parallel to the axis. Hence F_2 is also F_1; or the principal foci coincide. Again, we may say that the system is equivalent to coaxial surfaces separating media, of which the first and last indices are μ_0 and $-\mu_0$. Hence the focal lengths are equal and opposite; and as F_1 and F_2 coincide, so also H and H' coincide, and N and N' coincide. The entire system is equivalent in all respects to a spherical mirror, vertex at H, and centre at N.

The case of light reflected at the second surface of a lens may be treated as an example of three spherical surfaces, vertices A_1, A_2, A_3, separating media of indices 1, μ, $-\mu$, -1.

Hence if ρ and σ be the radii of curvature of the surfaces, taken positive if they are concave to the incident light as in the figure, we have

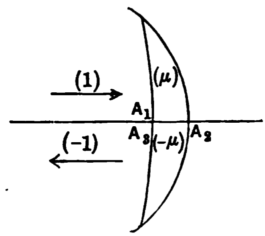


Fig. 51.

$$\kappa_1=(\mu-1)/\rho,\ \kappa_2=(-\mu-\mu)/\sigma,\ \kappa_3=(-1+\mu)/\rho,$$

$$\overline{A_1A_2}=t/\mu,\ \overline{A_2A_3}=(-t)/(-\mu),\ \overline{A_1A_3}=2t/\mu.$$

The sign of ρ must remain the same in κ_3 as in κ_1, for though the light is really reflected from A_2 towards A_3, we must treat it as if it moved onwards, and were incident on a third concave surface. For the same reason the distance A_2A_3 is to be treated as virtual and equal to $-t$. If then $t/\mu=\tau$, and if u be the distance of Q in front of A_1, we have by (I) Art. 75,

$$\begin{aligned}-a_3/a_0&=1+\kappa_1 u+\kappa_2(u+\tau)+\kappa_3(u+2\tau)\\&\quad+\kappa_1\kappa_2 u\tau+\kappa_1\kappa_3 u\,.\,2\tau+\kappa_2\kappa_3(u+\tau)\,\tau\\&\quad+\kappa_1\kappa_2\kappa_3 u\tau^2\\&=1+(\kappa_2+2\kappa_3+\kappa_2\kappa_3\tau)\,\tau\\&\quad+(\kappa_1+\kappa_2+\kappa_3+\kappa_1\kappa_2\tau+2\kappa_1\kappa_3\tau+\kappa_2\kappa_3\tau+\kappa_1\kappa_2\kappa_3\tau^2)\,u,\end{aligned}$$

in which the coefficient of u is the power.

Since $\kappa_3=\kappa_1$, this may be written as

$$-a_3/a_0=1+(2\kappa_1+\kappa_2+\kappa_1\kappa_2\tau)\,\{\tau+(1+\kappa_1\tau)\,u\}.$$

Now by Helmholtz's formula, $-l'a_3=la_0$; so that, by putting $l'=l$, $a_3=-a_0$, we see that the vertex of the equivalent mirror is at a distance $\tau/(1+\kappa_1\tau)$ behind A_1. This is equal to $t/\{\mu+(\mu-1)\,t/\rho\}$.

The centre of the equivalent mirror is the nodal point, and its distance in front of A_1 is found by putting $a_3=a_0$.

APPARENT DISTANCE.

85. When a small object is viewed by means of an optical system, symmetrical about an axis, the eye being on the axis of the system, and the object being perpendicular to the axis, the angle which it appears to subtend at the eye will depend on the positions of both eye and object.

The distance from the eye at which the object must be placed to subtend the same angle, when viewed directly, that it appears to subtend when seen through the instrument is called its apparent distance.

The angular magnification of any object depends on its apparent distance, for if E be the eye, PQ the object and $E\Pi$ the apparent distance, the angular magnification, *i.e.* the ratio of the angle which the object appears to subtend to the angle it subtends when viewed directly in the same position, is equal to

$$\frac{PQ}{E\Pi}\bigg/\frac{PQ}{EQ} = EQ/E\Pi.$$

When the image formed by the system is inverted, $E\Pi$ will be negative.

86. The value of the apparent distance for a system of thin lenses is calculated in the following manner:—

Let the lenses meet the axis in the points A_1, $A_2 \dots A_n$ in succession, and let E and Q lie outside A_1A_n. [In any case EA_1 and A_nQ can be regarded as algebraic quantities.]

Any one of the n lenses divides the distance EQ into two parts; any two of the lenses divide it into three consecutive parts; any three of the lenses divide it into four consecutive parts, and so on; lastly the n lenses divide it into $(n+1)$ consecutive parts.

The product of any parts that make up EQ, if divided by the (algebraic) focal lengths of the lenses at the points of division, gives a length. The algebraic sum of all such lengths together with EQ is the apparent distance $E\Pi$.

Let a ray from E cross the lenses at heights $y_1, y_2 \dots y_n$, and cross the object at height η_n; and let the successive angles of divergence be $\alpha_0, \alpha_1, \alpha_2 \dots \alpha_n$. Let the powers of the lenses be $\kappa_1, \kappa_2 \dots \kappa_n$.

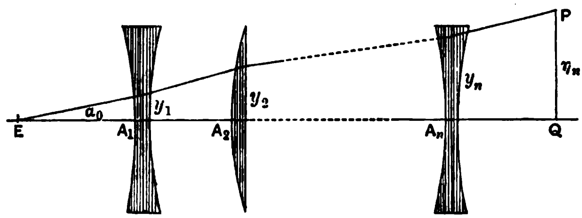


Fig. 52.

Then the length η_n of the object appears to subtend an angle α_0 at the eye, and therefore the apparent distance is η_n/α_0.

We can form the equations

$$\left.\begin{aligned} y_1 &= \alpha_0 \,.\, EA_1 \\ \alpha_1 - \alpha_0 &= \kappa_1 y_1 \\ y_2 - y_1 &= \alpha_1 \,.\, A_1A_2 \\ &\dots\dots\dots\dots \\ y_n - y_{n-1} &= \alpha_{n-1} \,.\, A_{n-1}A_n \\ \alpha_n - \alpha_{n-1} &= \kappa_n y_n \\ \eta_n - y_n &= \alpha_n \,.\, A_nQ \end{aligned}\right\} \dots\dots\dots\dots\text{(i)}.$$

Substitute from the first and second of these equations in the third, we have

$$y_2/\alpha_0 = EA_2 + \kappa_1 EA_1 \,.\, A_1A_2.$$

This is the apparent distance of A_2, when viewed from E through one lens, and is of the type enunciated.

Again, by substituting in the fourth and fifth equations we can determine y_3/α_0, and so on, and finally obtain α_n/α_0 and η_n/α_0 by an induction similar to that used in Art. 65. But we shall adopt a slightly different induction, by which η_n/α_0 is obtained directly.

Assume that the form given for the apparent distance holds good when any object is viewed through $(n-1)$ lenses. In that case the apparent distance of Q would have been calculated from the above equations with the exception of the last three and with the additional equation

$$\eta_{n-1} - y_{n-1} = A_{n-1}Q \,.\, \alpha_{n-1}.$$

Hence $\eta_n - \eta_{n-1} = y_n - y_{n-1} + A_nQ \cdot a_n - A_{n-1}Q \cdot a_{n-1}$

$$= A_nQ(a_n - a_{n-1})$$

$$= \kappa_n A_nQ \cdot y_n.$$

On dividing this equation by α_0, we see that the apparent distance of Q when seen through n lenses exceeds its apparent distance when seen through $(n-1)$ by the product of $\kappa_n A_n Q$ and the apparent distance of A_n seen through $(n-1)$ lenses. But the only terms in which the form assumed for the case of n lenses exceeds that assumed for $(n-1)$ lenses are those obtained by taking one of the points of section of EQ at A_n; each of these terms has $\kappa_n A_n Q$ as a factor, and the other factors of any term are obtained by taking all possible points of section in EA_n. But the sum of these terms is exactly the apparent distance of A_n when seen through $(n-1)$ lenses.

Hence if η_{n-1}/α_0 and y_n/α_0 have the form assumed, so also has η_n/α_0.

But we have seen that the form assumed holds for one lens and any position of the object; hence it is true generally.

87. If K denote the power of the system of lenses, u the distance of the eye from the eye-glass A_1, and v the distance of the object from the object-glass A_n, both measured away from the system, then the apparent distance is equal to

$$uvK + u\frac{\partial K}{\partial \kappa_n} + v\frac{\partial K}{\partial \kappa_1} + \frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n}.$$

This expression may be obtained by writing $E\Pi$ at length and making the substitutions $EA_r = u + A_1A_r$, and $A_rQ = A_rA_n + v$.

But it is obvious from equations (i) Art. 86 that y_n/α_0, α_n/α_0 and η_n/α_0 are the numerators of the last three convergents to the continued fraction

$$u + \frac{1}{\kappa_1 +}\,\frac{1}{A_1A_2 +}\cdots\frac{1}{\kappa_n +}\,\frac{1}{v}.$$

As shewn in Art. 64, the value of α_n/α_0 is $uK + \dfrac{\partial K}{\partial \kappa_1}$, and y_n/α_0 is the coefficient of κ_n in this, and may therefore be written as

$$\frac{\partial}{\partial \kappa_n}\left(uK + \frac{\partial K}{\partial \kappa_1}\right).$$

Hence $$\eta_n/\alpha_0 = v\alpha_n/\alpha_0 + y_n/\alpha_0$$

$$= uvK + v\frac{\partial K}{\partial \kappa_1} + u\frac{\partial K}{\partial \kappa_n} + \frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n}.$$

88. If two points on the axis be conjugate to each other in the refracting system, the apparent distance of either as seen from the other is plainly zero.

This affords the simplest way of expressing the conjugate property of two points without calculating the positions of the cardinal points in a system of finite power, or of the centre in a telescopic system.

Equating to zero the value of the apparent distance given in Art. 87, and comparing the equation with the relation

$$\left(uK + \frac{\partial K}{\partial \kappa_1}\right)\left(vK + \frac{\partial K}{\partial \kappa_n}\right) = 1,$$

contained in Art. 64, we may deduce the identity

$$\frac{\partial K}{\partial \kappa_1}\frac{\partial K}{\partial \kappa_n} - K\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} \equiv 1$$

given in Art. 67.

89. Apparent distance of an object when viewed through any coaxial refracting surfaces.

If the object and the eye do not lie in media of the same refractive index, we must understand by the apparent distance the distance of the object when placed in the same medium as the eye and viewed directly under the same angle as it appears to subtend through the system of refracting surfaces.

It is clear that we can form equations of exactly the same type as those in Art. 86 by using the equivalent thickness of each medium.

The apparent distance will therefore be

$$\mu_0 \left[\overline{EQ} + \Sigma \kappa_r \kappa_s \ldots \kappa_z \overline{EA_r} \,.\, \overline{A_r A_s} \ldots \overline{A_z Q}\right],$$

using the notation of the equivalent thickness.

The case of an object in air viewed through thick lenses may be included in this; or it may be brought more directly within the scope of Cotes's formula by omitting the distance HH' in each lens.

For example, the apparent distance of Q when seen through a thick lens, whose surfaces meet the axis in A and B and are of powers κ_1 and κ_2, may be written either as

$$(EA + AB/\mu + BQ) + \kappa_1 EA\,(AB/\mu + BQ) + \kappa_2 (EA + AB/\mu)\,BQ$$
$$+ \kappa_1\kappa_2 EA\,.\,AB/\mu\,.\,BQ,$$

or as

$$EH + H'Q + K\,.\,EH\,.\,H'Q,$$

making use of the unit points H, H' and the power K.

90. *A Galileo's telescope in normal adjustment is used to view an object at a given distance from the eye; shew that the angular magnification is greatest when the eye-glass is in contact with the eye, and least when the object-glass is in contact with the object. Shew also that an observer whose least distance of distinct vision is Δ cannot use the instrument for any objects within a distance $F + (\Delta - f)\,F^2/f^2$ from the object-glass.*

It is shewn in Chap. VI. that, if F and f be the *numerical* focal lengths of the (convergent) object-glass and (divergent) eye-glass of a Galileo's telescope in normal adjustment, their distance apart is $F-f$. Hence, if u be the distance of the eye behind the eye-glass, and v the distance of the object in front of the object-glass, the apparent distance of the object is

$$u+F-f+v+u\,(F-f+v)/f+(u+F-f)\,v/(-F)+u\,(F-f)\,v/(-Ff),$$

i.e. is

$$F-f+uF/f+vf/F.$$

Since $u+v$ is given and $F>f$, the apparent distance is least when u is zero and greatest when v is zero.

The angular magnification is therefore greatest when the eye is in contact with the eye-glass, and least when the object is in contact with the object-glass.

Again, if u' be the algebraic distance behind the eye-glass of the image of the object, the apparent distance of the object from that point is zero. Hence

$$-u'=vf^2/F^2+f\,(F-f)/F.$$

The necessary distance of the image in front of the eye, which is supposed at the eye-glass, being $>\Delta$, we must have $-u'>\Delta$ and $v>F+(\Delta-f)F^2/f^2$, for the distance of the object from the object-glass.

EXAMPLES.

1. A point on the first focal plane of a coaxial refracting system at distance y from the axis is the origin of a pencil of light; shew that the emergent rays make an angle y/f_2 with the axis.

2. The focal lengths of two coaxial systems are f_1, f_2 and f_1', f_2' respectively; c is the distance from the second focus of the first system to the first focus of the second; shew that the focal lengths of the combination are f_1f_1'/c and f_2f_2'/c, and the distance between its unit points or between its nodal points is $(f_1+f_2')(f_2+f_1')/c+F_1F_2'$, where F_1 and F_2' are the other foci of the systems.

3. Prove that if a coaxial refracting system be such that the distance between any two conjugate foci is constant, the linear magnification must be $\pm(\mu/\mu')^{\frac{1}{2}}$ for all points, where μ and μ' are the indices of refraction of the initial and final media.

4. When a small object is placed at A on the axis of a system of coaxial spherical refracting surfaces it is found that a real image is formed at B; at A and B two plane mirrors are placed at right angles to the axis. Prove that, if a small object be placed between A and the system, there will be formed between A and the system two series of real images, whose distances from A form harmonical progressions, and whose linear dimensions perpendicular to the axis are in the same ratio as their distances from A.

5. A pencil is refracted directly through any media separated by coaxial spherical surfaces, reflected at a spherical mirror on the same axis, and again refracted through the media in reverse order. Shew that the effect is the same as if the pencil had been reflected at a spherical mirror whose vertex and centre are the points conjugate in the refracting system to the vertex and centre respectively of the mirror. Shew that the focal length of the equivalent mirror is $ff_1f_2/(c^2-f^2)$, where f_1, f_2 are the focal lengths of the refracting system, f that of the mirror, and c the distance between its focus and the second focus of the system.

6. A thin lens, in which the index of refraction is μ, and the radii of whose surfaces are ρ and σ, is placed at distance d in front of a concave mirror of radius r, the medium between the lens and the mirror being of index μ'. Prove that, if $\mu'/(r-d)=(\mu-1)/\rho+(\mu'-\mu)/\sigma$, the effect of two refractions through the lens with an intermediate reflection is exactly equivalent to reflection at a plane mirror at distance $d(r-d)/\mu'r$ behind the lens.

7. A glass sphere of index μ and of radius $(\mu-1)c$ is cut in half and the two hemispheres are separated, the line joining their centres being perpendicular to their plane faces, which are opposite each other at distance c apart. The distances of two conjugate foci in front of and behind the first and second curved surfaces respectively being u and v, shew that the linear magnification is $(c-v)/(c-u)$.

8. Two equal hemispherical lenses of radius r and index μ are placed with their curved surfaces opposite each other at a distance c apart. Shew that the least distance between a real object and a real image is

$$c+2r\{1+1/\mu(\mu-1)\}.$$

Shew that if $c=2r/(\mu-1)$ the distance between any two conjugate foci is $\dfrac{\mu^2+1}{\mu}c$.

9. A pencil passes through a thick lens, being internally reflected once at each surface. Shew that the power of the lens is increased by

$$2\mu(\rho-\sigma)+t\,\{4(\mu-1)(\rho-\sigma)^2-2\rho\sigma(\mu+\mu^{-1})\}$$
$$-4t^2\mu^{-1}(\mu-1)(2\mu-1)\rho\sigma(\rho-\sigma)+4t^3\mu^{-1}(\mu-1)^2\rho^2\sigma^2,$$

where ρ and σ are the *curvatures* of the surfaces, t the thickness.

10. If (p, p'), (q, q'), (r, r') be three pairs of conjugate points on the axis of any coaxial refracting system, prove that

$$\begin{vmatrix} pq, & pr \\ p'q', & p'r' \end{vmatrix}^2=\frac{pq\,.\,qr\,.\,pr\,.\,p'q'\,.\,q'r'\,.\,p'r'}{f_1f_2}$$

where pq denotes the distance between the points p, q, with similar meanings for the other quantities, and f_1 and f_2 are the focal lengths of the system.

11. Prove that in any optical instrument in air the ratio of the apparent distance of any object to its distance from the point conjugate in the instrument to the eye is equal to the linear magnification for that point.

Shew that the apparent distance of Q as seen from E may be written either in the form $(EF_1\,.\,F_2Q-f^2)/f$ or as $EH+H'Q+EH\,.\,H'Q/f$, where F_1, F_2 are the principal foci, H, H' the unit points and f the focal length of the system.

Deduce from these forms and Cotes's formula for the apparent distance, the positions of the cardinal points and the formulæ for linear magnification.

12. In any coaxial refracting system in air the apparent distance of an object is equal to the focal length if either (i) the eye be at the first focus of the system, or (ii) the object be at the second focus.

The distance which an object must be moved to appear of the same size when viewed directly and when viewed through the instrument is independent of the position of the object if the eye be at the first unit point, and is independent of the position of the eye if the object be at the second unit point, the direction of measurement being from the eye to the object.

13. An object at a given distance from the eye is viewed directly through a sphere of glass; determine the range of values of the angular magnification for all positions of the sphere, distinguishing the cases in which the distance $>$ or $<$ 6 (radius of the sphere).

14. Three convergent lenses of numerical focal lengths f, g, h are placed on the same axis at distances $f+2g$, $h+2g$ apart. Shew that, if u and v be the distances of two conjugate foci measured outwards from the extreme lenses,

$$u/f^2+v/h^2=1/f+1/g+1/h,$$

and determine the linear magnification.

15. The distance of an object from a system of n thin lenses each of power κ and at distance a apart is v, and the distance of the eye from the system is u, both measured away from the system; shew that the apparent distance is

$$\{(\kappa uv+u+v)\sin n\theta-(u+v-a)\sin(n-1)\theta\}/\sin\theta,$$

where $$\sin^2\tfrac{1}{2}\theta=-\tfrac{1}{4}a\kappa.$$

16. Shew that, if u be the distance of the eye from a system of n thin lenses of focal lengths f_1, $f_2 \ldots f_n$, at distances a_1, $a_2, \ldots a_{n-1}$ apart, the apparent distance of an object at distance v from the system is equal to

$$a_1a_2\ldots a_{n-1}uv \begin{vmatrix} 1/u+1/f_1+1/a_1, & 1/a_1, & \ldots\ldots & 0 \\ 1/a_1, & 1/a_1+1/f_2+1/a_2, & 1/a_2, \ldots\ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 \ \ldots\ldots & 1/a_{n-2}, & 1/a_{n-2}+1/f_{n-1}+1/a_{n-1}, & 1/a_{n-1} \\ 0 \ \ldots\ldots & & 1/a_{n-1}, & 1/a_{n-1}+1/f_n+1/v \end{vmatrix}$$

17. A spherical shell, whose internal and external boundaries are of radii a and b, is composed of heterogeneous material, whose index of refraction is a function of the distance r from the centre. The cavity is filled with a homogeneous medium of index μ', and the shell is in air. Shew that for small pencils passing directly this system of refracting surfaces may be replaced by a thin lens at the centre, of power in air equal to

$$2\left[\left(\frac{1}{\mu_b}-1\right)\frac{1}{b}+\left(\frac{1}{\mu'}-\frac{1}{\mu_a}\right)\frac{1}{a}+\int_a^b\frac{d\mu}{\mu^2 r}\right] \text{ or to } 2\left[\frac{1}{\mu' a}-\frac{1}{b}+\int_a^b\frac{dr}{\mu r^2}\right].$$

18. A heterogeneous medium has for its surfaces of equal refractive index a continuous series of surfaces of revolution about the axis of z. The medium extends from $z=z_0$ to $z=z_1$, and the extreme media are uniform and of indices equal to those of the heterogeneous medium at its boundaries. Prove that, if K and L be any two independent solutions of the equation $\mu\frac{d}{dz}\left(\rho\frac{dK}{d\mu}\right)=K$, the power of the system for small pencils passing near the axis of z is $(K_0L_1-K_1L_0)/C$; the distance of the first focus in front of the first surface is

$$\mu_0\rho_0\left(L_1\frac{dK_0}{d\mu}-K_1\frac{dL_0}{d\mu}\right)\Big/(K_0L_1-K_1L_0),$$

and of the second focus beyond the last surface is

$$\mu_1\rho_1\left(L_0\frac{dK_1}{d\mu}-K_0\frac{dL_1}{d\mu}\right)\Big/(K_0L_1-K_1L_0);$$

where the suffixes indicate the values at the boundaries, ρ is the radius of curvature at the vertex of the surface of index μ, and C is the constant quantity

$$\rho\left(K\frac{dL}{d\mu}-L\frac{dK}{d\mu}\right).$$

19. A system of refracting surfaces is composed of the further portions of spheres having a common tangent plane. The radii of the internal and external boundaries are a and b respectively, and the refractive index at any sphere is $2b/(\text{diameter})$. The external medium is uniform and of refractive index unity, and the internal medium is also uniform and of refractive index (b/a). Shew that for small pencils incident directly, the distance of the first principal focus in front of the external surface is $b\{\cot(\log b/a)-1\}$, and of the second behind the internal surface is $a\{\cot(\log b/a)+1\}$; and that the two focal lengths of the system are $-a\operatorname{cosec}(\log b/a)$ and $-b\operatorname{cosec}(\log b/a)$.

20. A system of refracting surfaces is composed of portions of spherical surfaces concave to the origin, such that the radius of curvature at distance z from the origin is z/n^2, and the refractive index there is κz. The system extends from $z=a$ to $z=b$ and is bounded by media of indices κa and κb. A small object is placed at the origin perpendicularly to the axis; shew that the distance $b-v$ of the image from the origin, and its linear magnification m are given by

$$v=b\tanh(n\theta+\lambda)\coth\lambda,\quad m=\operatorname{sech}(n\theta+\lambda)\cosh\lambda,$$

where
$$\theta=\log(b/a)\text{ and }n=\tanh\lambda.$$

21. A small object perpendicular to the axis is viewed through a heterogeneous medium, whose surfaces of equal refractive index are surfaces of revolution about the axis of z. The eye and the object are in uniform media of indices equal to those of the heterogeneous medium at its boundaries. Shew that, if u be the distance of the eye, and v the distance of the object from the boundaries of the medium, the apparent distance is

$$\left[\left(u\frac{dZ_0}{dz}-Z_0\right)\left(v\frac{dY_1}{dz}+Y_1\right)-\left(u\frac{dY_0}{dz}-Y_0\right)\left(v\frac{dZ_1}{dz}+Z_1\right)\right]\Big/\left(Y_0\frac{dZ_0}{dz}-Z_0\frac{dY_0}{dz}\right),$$

where Y and Z are any two independent solutions of the equation

$$\frac{d}{dz}\left(\mu\frac{dy}{dz}\right)=\frac{y}{\rho}\frac{d\mu}{dz},$$

and the suffixes indicate values at the boundaries.

CHAPTER VI.

OPTICAL INSTRUMENTS.

91. The Eye.

THE human eye is an optical instrument, containing different transparent media, separated by curved surfaces, by means of which an image of the object seen is formed on the layer of nerves at the back of the eyeball, whence the sensation is transmitted to the brain by the optic nerve. The eyeball is nearly spherical,

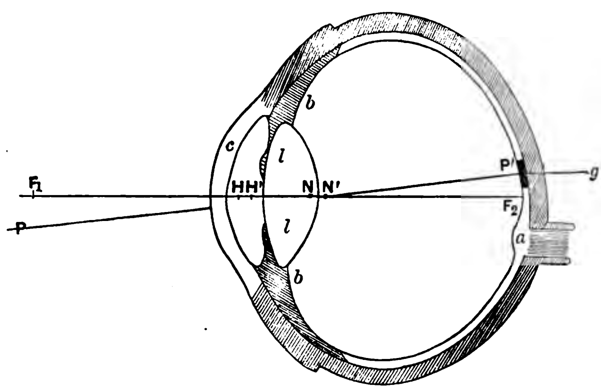


Fig. 53.

except in front, where it projects slightly. Fig. 53 is a horizontal section of the eye.

The outside covering of the eye is a thick white opaque membrane, called the *sclerotica*; this is the part ordinarily called the white of the eye. It is pierced at *a* by the optic nerve on the side nearer the nose. In the front of the eye its place is taken by

the *cornea*, *c*, which is a hard, transparent substance. Its outer surface is in the shape of a prolate spheroid, and is kept damp and smooth by the eye watering.

The interior of the sclerotica is lined by a softer membrane, called the *choroid*, which is attached to the sclerotica at the entrance of the optic nerve. At the edge of the cornea the choroid is continued into the *iris*, *b*, which is an opaque screen, with a circular aperture, the *pupil* of the eye. The front of the iris is coloured; its various tints in different persons are ordinarily described as the colour of the eyes.

Imbedded in the iris are two important muscles; one, the contractor muscle or sphincter, surrounds the pupil; the other, the dilatator, runs round the iris nearer its junction with the cornea. The function of these muscles is to alter the aperture of the pupil, which is largest in weak light, but contracts in strong light, varying in diameter from about $\frac{1}{4}$th to $\frac{1}{10}$th of an inch.

The *retina* is an extension of the optic nerve, covering the interior of the choroid. At the back of the eye and near the centre is a small yellow spot, *g*, with a slight hollow, called the *fovea centralis*; this is the most sensitive part of the retina, and an object is seen most clearly when the image falls there. The retina continues to the front of the eye, becoming thinner. Its construction is very complicated, each element, of about ·004 mm. diameter, consisting of rods, piles and cones. Light falling on a single element of the retina produces a single sensation; and the eye cannot separate two sources of light unless the central points of their images are formed on two elements separated by another element not receiving light or at least less light than the two others.

All the retina is sensitive to light, except the point of entry of the optic nerve, which is known as the blind spot.

The *crystalline lens*, *l*, is a transparent colourless body, placed behind the iris, and supported by the ciliary processes. It is shaped like a double convex lens, having the anterior surface less curved than the posterior; and it is fibrous in construction, the density increasing towards the centre. This serves to correct aberration by increasing the convergency of the central rays compared with the extreme rays. The substance of the lens is double refracting, and the front surface is said to be part of an oblate

spheroid and the back part of a paraboloid. Its total refractive index, which is the refractive index of a homogeneous lens of the same shape and power, can be found; the value is somewhat more than the refractive index of water. The space between the cornea, the iris and the lens is filled with the *aqueous humour*. This is a clear transparent fluid, and its refractive index does not differ appreciably from that of water.

The space between the lens and the retina is filled with the *vitreous humour*, which is a transparent fluid, slightly viscous, and with a refractive index just exceeding that of water. Both humours are enclosed in transparent membranes of extreme fineness and delicacy.

92. Cardinal Points, and Accommodation.

When light enters the eye, refraction takes place at the surfaces of the cornea and of the crystalline lens. The centres of these surfaces lie on a straight line, called the optic axis, meeting the retina between the optic nerve and the fovea centralis. The radii of curvature of the surfaces are different for different people, and in the same person alter with age, and also alter according to the distance of the object viewed. Hence the positions of the cardinal points and the focal lengths vary; but clear images of objects within certain extreme distances of distinct vision can be formed on the retina. This power of adjustment of the eye, by which we can see successively objects at different distances, is known as *accommodation*. The smallest distance of an object which can be seen clearly is usually 5 or 6 inches; the greatest is generally so large that it is usual to suppose that a normal eye, when relaxed, receives parallel rays naturally on the retina, but accommodates itself for rays diverging from nearer objects.

The accommodation of the eye is effected chiefly by an alteration, produced by muscular effort, in the front surface of the crystalline lens. This becomes more convex and approaches the cornea.

The following values of the constants of the eye are given by Helmholtz*, from which the positions of the cardinal points, and the focal lengths are calculated.

* Helmholtz, *Physiologische Optik*, 1896, p. 140.

	Distant object	Near object 15 cm. distance
Index of refraction of the humours	1·3365	1·3365
" " " crystalline lens	1·4371	1·4371
Radius of the outer surface of the cornea	7·8 mm.	7·8 mm.
" " first lens surface	10 "	6 "
" " second "	6 "	5·5 "
Distance of the first lens surface from the cornea	3·6 "	3·2 "
Distance of the second lens surface from the cornea	7·2 "	7·2 "
First focal length	15·5 mm.	14 mm.
Second "	20·7 "	18·7 "
Distance of first unit point behind the cornea	1·75 "	1·9 "
" second " " " "	2·1 "	2·3 "
Distance of the first nodal point behind the cornea	7·0 "	6·6 "
Distance of the second nodal point behind the cornea	7·3 "	7·0 "
Distance of the first focus in front of the cornea	13·7 "	12·1 "
Distance of the second focus behind the cornea	22·8 "	21·0 "

The refraction from the cornea into the aqueous humour is disregarded, owing to the very small difference there is between their refractive indices; and the aqueous humour is considered as extending to the outer surface of the cornea; the thickness of the cornea is only ·4 mm.

The distance between the unit points, or between the nodal points, lies between ·3 and ·4 mm., according to the adjustment of the eye; and the image formed is exactly the same as would be formed by a single refracting surface, with its vertex at H and centre at N, the index of refraction being that of the humours, and the retina being brought forward the small distance HH'. The positions of these points when the eye is viewing a distant object are marked in Fig. 53.

The line PN is the *line of sight* for the object P, and is parallel to the line $N'P'$; the apparent angle between two objects, being estimated by the distance between their images on the retina, is the angle between the lines from N' to their images on the retina, which is also the angle between their lines of sight. Hence in any statement as to the position of the eye it must be understood that it is N which occupies a definite position.

93. Field of View.

A small object is seen most distinctly when its image is formed on the fovea centralis; but there will be a conscious sensation of light, if light fall on any other part of the retina. With the axis of the eye in a definite direction, there is therefore a wide field of view, extending sometimes to nearly 90° from the axis, which is seen vaguely, and in which any sudden bright change is at once noticed. The distinct part of the field does not extend to more than 5° from the axis, but the eye can turn in its socket through an angle of 55° in every direction; and therefore by the rapid motion of the eye the distinct field of view is largely extended. The persistence of the impression on the retina also aids in forming the connected idea of a wide view. A bright light produces a sensation lasting $\frac{1}{24}$th of a second, a weak light one lasting $\frac{1}{10}$th of a second. The eye has the greatest field of view of any optical instrument.

The separating power of the eye is measured by the smallest angle between two bright points, which yield separate images. That this limit exists is due, as stated above, to the construction of the retina in elements yielding only one sensation; and the limit is, for single origins of light as stars, about 40″. For extended objects however this limit is increased by aberrations to about 1′; and it is entirely unnecessary to pay any attention to the defects of optical images, mathematically calculated, which do not exceed this limit.

Again, this construction of the retina enables objects at different distances to be seen simultaneously. When the eye is accommodated for an object at a certain distance, the pencils of rays from points at other distances will meet the retina in small *circles of diffusion*; and these, if small enough, are perceived as points. It is stated that an eye, adjusted for rays from infinity, can also see clearly objects more than 25 yards distant.

The size of the circles of diffusion is decreased if the effective aperture of the pupil be decreased; thus looking through a pin-hole enables us to see objects much nearer than is possible to the unassisted eye; and short-sighted people also habitually contract the opening of the pupil.

The estimate that we form of the distance of an object is a psychological problem. It seems to be principally aided by the

fact that we have two eyes. On each retina there are corresponding points; *i.e.* points such that images formed on them yield only a single sensation to the brain; and vision is clear from both eyes, when the optic axes of the eyes are so directed that the two images of the same origin fall on corresponding elements. The muscular sensations produced in moving the eyeballs for this purpose are partly the basis of the judgment formed of distance.

The same may be said of the muscular movements by which accommodation is produced; all these changes are necessarily much larger for near objects than for distant objects; and no estimate whatever can be formed of the actual distance of unknown objects removed beyond a certain distance. Part also of the judgment of distance may arise from the apparent size of the images of known objects.

Again, when a distant object is viewed, the optic axes of the two eyes are parallel, and the images of each point of the object will be formed on the corresponding elements of the retinas. But when the object is near, the optic axes are convergent, and therefore the images on the retinas of an extended object cannot be identical. Hence, and from the effects of light and shade, arises the idea of relief, that we attach to such objects. The principle is exemplified in the stereoscope, where two plane pictures, taken from slightly different points of view, and seen one by each eye, give the idea of a solid object.

94. Limiting distances of distinct vision.

The distances, within which vision is clear by the power of accommodation of the eye, are very different in different eyes. The normal or *emmetropic* eye is supposed to be in its natural relaxed state when parallel rays are brought to a focus on the retina, and to accommodate itself actively for objects between infinity and a certain minimum distance of about 6 inches.

But in a short-sighted or *myopic* eye both the maximum and minimum limits are much smaller; distinct vision extends only from a point, often quite near the eye, to a superior limit, never very great. In fact, if this superior limit be as much as 10 yards, vision is appreciably clear to infinity. Since in the myopic eye the image of an object at a finite distance falls naturally on the retina, parallel rays come to a focus before the retina, and the depth of the myopic eye is greater than that of the emmetropic eye.

In a long-sighted or *hypermetropic* eye the minimum distance of distinct vision is greater than in the normal eye, and, when the eye is relaxed, the rays must possess a certain convergence to be brought to a focus on the retina. The range of distinct vision extends through infinity to a negative value; and in isolated cases both limits may even be negative. A hypermetropic eye is always in a state of tension when looking at natural objects; and it is difficult to determine experimentally the limits of vision, as the accommodation of the eye becomes permanent. In its relaxed state the depth of the hypermetropic eye is less than that of the normal eye.

Both myopia and hypermetropia can be corrected by the use of spectacles.

The distances u and v of an object and its image from a thin lens are connected by the equation $1/v - 1/u = 1/f$. If the thickness of the lens be taken into account, these distances, u and v, are measured from the first and second unit points of the lens respectively.

Let a and b be the least and greatest distances from the lens for distinct vision with the naked eye. These are less than the distances from the first nodal point N of the eye by the distance between the lens and that point, usually about $\frac{1}{2}$ inch, which can be disregarded in comparison with the other quantities involved, except in the case of strong lenses.

To make the greatest distance of vision with the lens infinity, we put $u = \infty$, $v = f = b$; the focal length of the lens required is therefore the greatest distance of distinct vision, or the distance with the eye relaxed. The least distance of distinct vision with the lens is then found by putting $f = b$, $v = a$, and is therefore $ab/(b - a)$.

Thus a short-sighted eye, for which b is positive, will require a divergent lens, usually double-concave, which will increase the lower limit of distinct vision. For example if the natural range of vision be 3 to 6 inches, a divergent lens, focal length 6 inches, will make the range 6 inches to infinity.

A long-sighted eye, for which b is negative, will need a convergent lens, which will decrease the least distance of distinct vision. For example if the natural range of vision be from 12 inches through infinity to -12 inches, a convergent lens of focal length 12 inches gives the normal range.

In all eyes the least distance of distinct vision increases during life. This decrease in the power of accommodation is known as *presbyopia*. For an eye that enjoyed normal vision it may be necessary to use a weak convergent lens for reading and writing.

95. Astigmatism of the eye.

In many eyes the refracting surfaces are not truly spherical. In consequence of this defect the rays from a single origin of light do not come to a focus, but pass through two small focal lines, and meet the retina in a small area. If this area be small enough the origin of light may be seen distinctly; but when an extended object is viewed, though horizontal lines, for example, may form clear images on the retina, the vertical lines of the object may form blurred images. This defect of the eye is known as *astigmatism*, and may be remedied by the use of lenses with cylindrical instead of spherical faces.

TELESCOPES.

96. A telescope consists of a system of lenses arranged on the same axis and used for the purpose of viewing distant objects, especially the stars and other celestial bodies. In reflecting telescopes some of the lenses may be replaced by mirrors, but the principles remain the same in all cases.

The aim of a telescope is twofold; first, to increase the angle between pencils from distant sources of light so that the eye can distinguish between them; secondly, to bring into the eye an increased quantity of light from objects too faint to be perceived by the unassisted eye.

The lens on which the light first falls is called the *object-glass*; its image formed in the rest of the telescope is known as the *eye-ring*. All the light that falls on the object-glass will finally pass through the eye-ring; and the eye should therefore, if the eye-ring lie behind the telescope, be placed there in order that as much light as possible may enter the eye.

If the eye-ring fall within the telescope, the eye is placed close to the eye-glass, and the effective aperture of this lens will in this case not exceed that of the pupil of the eye.

The ray which finally crosses the axis at the first nodal point of the eye is called the *principal ray*; and the final image of an origin of light off the axis must be considered to be formed on this principal ray, or in its direction if the instrument be in normal adjustment, when the image of a very distant origin of light will itself be very distant.

The image of the eye-glass in the previous part of the telescope serves as a rule to determine the field of view, and has been called the *entrance-pupil.*

97. Magnifying power.

When two objects at unknown distances, or at known distances that cannot be varied, are compared, the mind will, apart from other reasons, judge that one to appear the larger, whose image occupies the larger area on the retina; and the ratio of their apparent magnitudes will be that of the angles which they subtend at the eye. In the same way if any object be viewed directly and also through a telescope, the magnification produced is estimated by the ratio of the angle which the image subtends at the eye to the angle which the object subtends when viewed directly in its actual position.

This ratio depends on the position of the object as well as that of the eye (cf. Art. 85); but the telescope is certainly used for objects at a considerable distance, and therefore to compare one telescope with another we define the *magnifying power of a telescope as the angular magnification of a very distant object.*

The origins of the rays may be practically at infinity, as in the case of any celestial objects; and the magnifying power is therefore the ratio of the apparent angular interval between two stars to their angular interval when viewed directly.

First, let the instrument have a finite focal length, and let the eye be accommodated for a finite distance (cf. Fig. 54). The image of a very distant source of light on the axis will then be formed at the second focus F_2 of the instrument, and if the eye be accommodated to this distance of the image, the rays will then be focussed on the retina. Again, the rays from a distant source of light, σ, lying off the axis, are brought to a focus at σ' on the second focal plane; and if E be the first nodal point of the eye, the apparent angular interval is α', where $\alpha' = \sigma' E F_2$.

The incident rays from σ are parallel to each other and make an angle α with the axis; while the incident ray corresponding to the emergent ray $\sigma' E$ crosses the axis in E', the point conjugate to E in the instrument.

Since E is necessarily at a finite distance from F_2, the conjugate E' is a finite point, and the two distant sources of light subtend the same angle when viewed directly from E' as from E.

The magnifying power, M, is therefore defined as α'/α, which is the reciprocal of the linear magnification for the point E', and is therefore equal to f/Δ, where f is the focal length of the

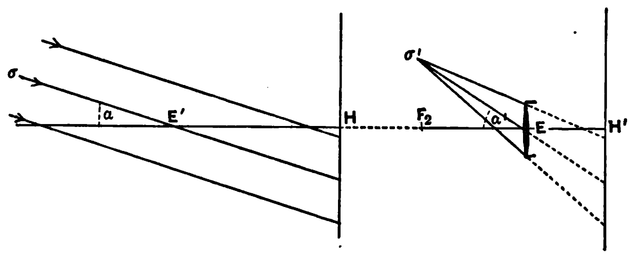


Fig. 54.

telescope, and Δ is the distance of vision, EF_2, to which the eye is accommodated.

In any given arrangement of the telescope we may also express the magnifying power by Cotes' formulæ, in terms of the positions of E' or E.

If A_1 denote the object-glass, A_n the eye-glass, then (i) in the case when the eye is at the eye-ring, E' is the object-glass; and (I) Art. 65 gives

$$M_1 = (\text{radius of object-glass})/(\text{radius of eye-ring})$$

$$= 1 + \sum_2^n \kappa_r \kappa_s \ldots A_1 A_r \,.\, A_r A_s \,.\, \ldots;$$

(ii) when the eye is at the eye-glass, then E' is the entrance-pupil, and (II) Art. 65 gives

$$M_2 = (\text{radius of entrance-pupil})/(\text{radius of eye-glass})$$

$$= [1 + \sum_{n-1}^1 \kappa_t \kappa_s \ldots A_n A_t \,.\, A_t A_s \,.\, \ldots.]^{-1}.$$

Secondly, when the instrument is in normal adjustment (Art. 72) and parallel rays are focussed naturally on the retina, the

principal incident ray will cross the axis at E' and enter the eye at E, as before, but the emergent pencil will consist of parallel rays as well as the incident pencil. The magnifying power is still expressed as α'/α. In this case however, and in this only, it is independent of the position of the eye, and either expression, M_1 or M_2, may be used. As we have seen in Art. 72, it is the reciprocal of the constant linear magnification produced by the system of lenses in normal adjustment, and is therefore equal to $-F/f$, where F and f are the (finite) focal lengths of any two parts into which the system may be divided.

In all actual telescopes, when normally adjusted, the position of the eye differs very slightly from the *optical centre* of the telescope, and the angular magnification of any finite object is therefore practically equal to the magnifying power as defined above.

98. Field of View.

The field of view of a telescope is that part of space, symmetrical about the axis, which can be seen through it. It is measured by the semi-vertical angle of the cone, which for points at some distance from the telescope forms its boundary.

The light from any object in front of the telescope fills the object-glass, and part of this will in passing through the telescope traverse the various lenses and also, it may be, diaphragms placed to diminish aberration, and finally enter the pupil of the eye.

Let O_1O_2 be the diameter of the object-glass, and D_1D_2 that of the image of any one of the lenses or diaphragms formed by the part of the telescope from the object-glass up to that lens or diaphragm; then four cases, as shewn by the figures, may arise according as D_1D_2 is in front of or behind the object-glass, and as it is greater or less than the object-glass.

Of all the rays that an object sends to the object-glass only those (produced to be virtual if necessary) which meet the plane of the diaphragm-image within D_1D_2, will ultimately pass within the diaphragm; and it is clear from the Figures 55 that if the object lie within the outer region shaded dark, the rays are entirely lost during their passage through the telescope; if it lie in the

region shaded lightly, then part are lost, while if it lie in the

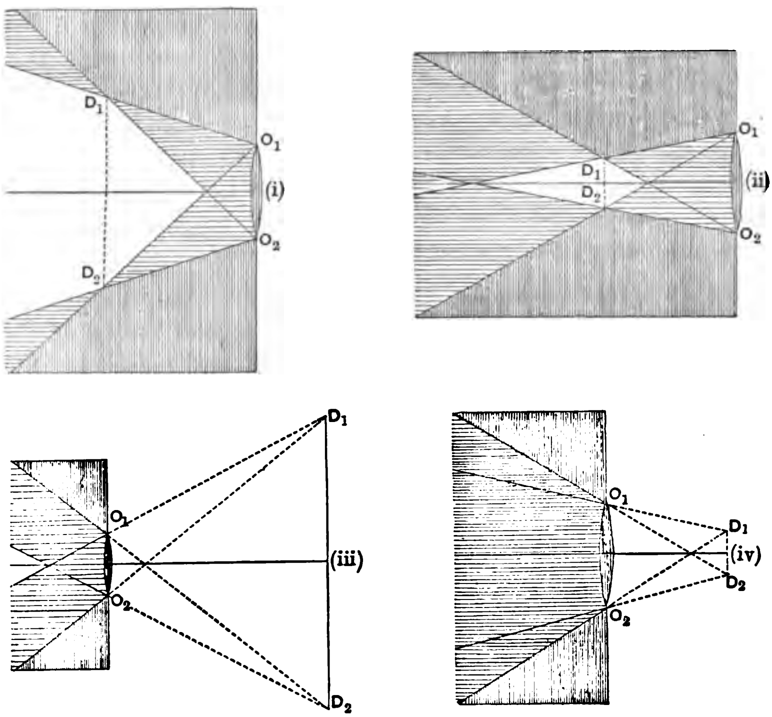


Fig. 55.

unshaded region, the whole of the pencil that fills the object-glass passes within the diaphragm.

If all the diaphragm-images be constructed, the extreme field of view *for distant objects* is measured by the least value obtained for the angle made by D_1O_2 with the axis. The field of view seen by pencils that fill the object-glass is, *again for distant objects,* measured, in cases (i) and (iii) by the least value of the angle made by D_1O_1 with the axis, while in cases (ii) and (iv) no distant part of the field can be seen by pencils that fill the object-glass.

We must further ensure that all the rays that fill the object-glass and emerge from the instrument can enter the pupil of the eye. The image of the pupil must be taken as one of the diaphragm-images. When the eye is at the eye-ring, this

image coincides in position with the object-glass; and all these rays will enter the pupil only if the eye-ring be less than the pupil. When this latter condition is not fulfilled, the aperture of the object-glass is practically diminished in the ratio *pupil* : *eye-ring*, but such failure to enter the pupil on the part of all the rays would not occur when large magnifying powers are used, and it also sacrifices one chief object of an astronomical telescope, that of collecting light.

When the eye is at the eye-glass, let D_1D_2 represent the image of the pupil of the eye in the telescope. This is now the entrance-pupil. Then we see that in cases (i) and (iii) the pencil of rays which on incidence fills the object-glass will not on emergence completely fill the pupil of the eye; while on the contrary, in cases (ii) and (iv), the pencils which on emergence fill the pupil will be parts only of the light incident on the object-glass.

This latter statement does not apply to the small unshaded region in case (ii), which can be seen by full pencils.

The opera-glass (Art. 105) is an example of cases (iii) and (iv); the astronomical telescope of case (i).

99. Angular radius of the field of view.

Let the powers of the lenses, reckoning from the object-glass to the eye-glass, be $\kappa_1, \kappa_2 \ldots \kappa_n$, and let K_n be the power of the system. Let $y_1, y_2 \ldots y_n$ be the radii of the lenses. (The term "aperture" is generally used in astronomical works for their diameter.)

The position of the entrance-pupil D is given by the equation (Art. 67),

$$0 = DA_1 \cdot \frac{\partial K_n}{\partial \kappa_n} + \frac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n}.$$

The radius of the entrance-pupil is $y_n \Big/ \dfrac{\partial K_n}{\partial \kappa_n}$. If this be negative, the entrance-pupil is an inverted image. Hence supposing the field of view governed by the eye-glass, the angular radius of the field of view seen by pencils that fill the object-glass is $\left\{y_n - y_1\left(\dfrac{\partial K_n}{\partial \kappa_n}\right)\right\} \Big/ \left(\dfrac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n}\right)$, and that of the extreme field is $\left\{y_n + y_1\left(\dfrac{\partial K_n}{\partial \kappa_n}\right)\right\} \Big/ \left(\dfrac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n}\right)$, where the brackets indicate that only the numerical values of the differential coefficients are to be substituted.

This assumes that the other lenses of the telescope are large enough to permit the full pencil to pass.

We can calculate their *effective apertures* as follows. Let Θ denote the angle of the field of view; the distance u from the object-glass of the vertex of the cone bounding it is given by $y_1 = \Theta . u$, and then the apparent distance of the rth lens as seen from this point is equal to η_r/Θ, where η_r is the effective aperture.

Using the formula for apparent distance (Art. 87) we obtain

$$\eta_r = y_1 \frac{\partial K_r}{\partial \kappa_r} \mp \Theta_1 \frac{\partial^2 K_r}{\partial \kappa_1 \partial \kappa_r},$$

or

$$\eta_r' = y_1 \frac{\partial K_r}{\partial \kappa_r} \pm \Theta_2 \frac{\partial^2 K_r}{\partial \kappa_1 \partial \kappa_r},$$

according as we use the aperture for full pencils Θ_1, or the extreme aperture Θ_2, and the upper sign in the ambiguities is to be taken if the entrance-pupil lie in front of the object-glass.

If the actual radius of the rth lens be numerically less than η_r or η_r', then that lens determines the field of view; and we obtain similar values for Θ_1 or Θ_2 as before, with r in place of n.

These are the necessary radii to allow all rays from very distant origins to pass; but if we consider finite objects, then the necessary radii will be given by the following rule;—

Let z be the distance of the origin of light in front of the object-glass, and y its distance from the axis. Then every ray of the pencil from it that fills the object-glass will fall within the rth lens, if the radius of that lens exceed the numerical value of $y_1 \left(\frac{\partial K_r}{\partial \kappa_r}\right) + \frac{y + y_1}{z} \left(\frac{\partial^2 K_r}{\partial \kappa_1 \partial \kappa_r}\right)$, where K_r is the power of the system composed of the first r lenses.

100. It must not be supposed that the investigations of Geometrical Optics give a sufficiently accurate account of the appearances in the focal plane of an object-glass; according to these the image of a star is a definite point of light of no appreciable magnitude. The more certain deductions of Physical Optics, based on the Undulatory Theory and not on the fallacious rectilinear propagation of light, shew that the image of a star is really a small bright circle surrounded by fine bright rings, of which the outer ones are coloured.

The angle in seconds of arc which the radius of the circle subtends at the centre of the object-glass is given* as $(13''\cdot 7)/d$, where d is the diameter of the object-glass in centimetres.

As the intensity of the light falls off towards the edge of the circle, it is usual to consider that two stars can be separated if the distance between the centres of their circular images be greater than this angle, and if moreover the eye-piece can magnify the angle to be apparently greater than 1′. For instance whatever the magnifying power, a telescope of 36 inches aperture cannot separate stars nearer than ·15″; and with such a telescope a magnifying power less than 400 would be useless.

* Mascart, *Traité d'Optique*, Vol. I. p. 315 (1889).

101. The Astronomical Telescope.

In its simplest form this instrument consists of two convergent lenses on the same axis. The radius and the focal length of the object-glass are much greater than those of the eye-glass.

It is generally assumed that rays from a star emerge from the instrument as a pencil of parallel rays, so that they can be received without active accommodation by a normal eye. The telescope is then *normally adjusted* (Art. 72); and the necessary

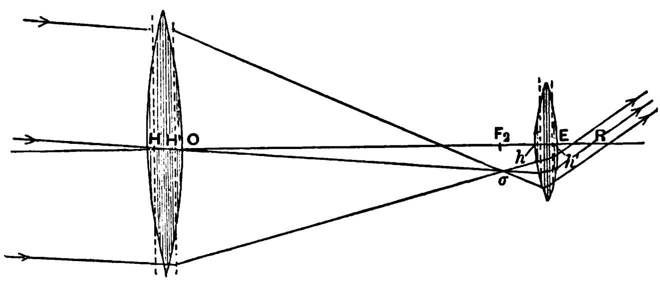


Fig. 56.

condition is that the second focus F_2 of the object-glass coincide with the first focus f_1 of the eye-glass.

If we trace the course of the pencil of rays from a star lying off the axis of the telescope, these are first brought to a focus σ on the common focal plane of the lenses, and then emerge from the eye-glass as a pencil of parallel rays.

First, treating the lenses as thin, the *principal* ray passes without deviation through the point O where the object-glass meets the axis, and after refraction through the eye-glass, enters the eye at R, which is the position of the eye-ring; the emergent rays are all parallel to σE, where E is the point in which the eye-glass meets the axis, since such a ray would pass through the eye-glass without deviation.

The magnifying power, defined as the ratio of the angles made by the emergent and incident rays with the axis, is therefore equal to the ratio of the angles σEO and σOE, *i.e.* to $-F/f$, where F and f are the focal lengths of the object-glass and eye-glass respectively. The negative sign implies that the apparent field of view is inverted. The apparent field of view is, in fact, turned through two right angles. In normal adjustment the magnifying power is independent of the position of the eye.

Secondly, if we take into account the thicknesses of the lenses, or the facts that the object-glass practically consists of two lenses, to render it achromatic, and that the eye-glass is replaced by an eye-piece of two or more lenses, then the incident rays are all parallel to the ray $H'\sigma$ from the second unit point of the object-glass to the focus σ, and the emergent rays are all parallel to the ray σh, where h is the first unit point of the eye-piece.

The magnifying power, in normal adjustment, is equal to the ratio of the angles made by σh and $H'\sigma$ with the axis, *i.e.* is $-F_0/f_e$, where F_0 and f_e are the focal lengths of the compound object-glass and of the eye-piece respectively.

102. Eye-ring and field of view.

The lenses being considered thin, let F_1 and F_2 be the principal foci of the object-glass, f_1 and f_2 those of the eye-glass; F_2 and f_1 coincide in normal adjustment, and the distance of the object-glass from f_1 is therefore F. Hence the eye-ring is at distance f^2/F behind f_2, and its radius, ρ, is $y_0 f/F$, where y_0 is the radius of the object-glass. The eye should be placed there to secure all the light that falls on the object-glass; and the tube of the instrument is usually prolonged beyond the eye-glass to the requisite distance $(f+f^2/F)$.

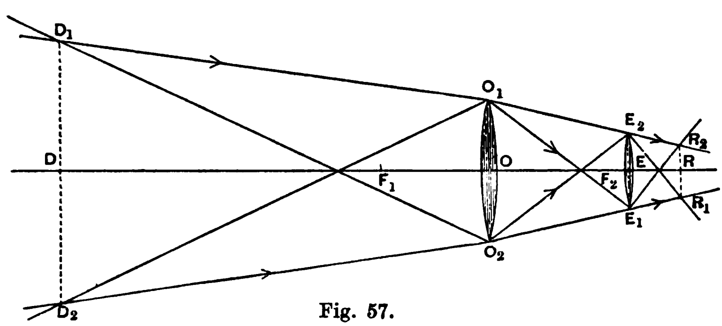


Fig. 57.

For the field of view, we construct the entrance-pupil D_1D_2, *i.e.* the image of the eye-glass in the object-glass. This lies at distance F^2/f in front of F_1, and is of radius $y_e F/f$, where y_e is the radius of the eye-glass. We assume that the entrance-pupil is greater than the object-glass; the eye-glass will then be larger than the eye-ring.

It is plain that the pencils from all points beyond D_1D_2 and

within the lines O_1D_1, O_2D_2 produced, which fill the object-glass, will traverse the eye-glass and emerge to fill the eye-ring.

The angular radius Θ_1 of the part of the sky thus visible is the angle made by O_1D_1 with the axis, which is equal to

$$\left(y_e\frac{F}{f}-y_o\right)\Big/\left(F+\frac{F^2}{f}\right).$$

Hence $$\Theta_1=\frac{y_eF-y_of}{F(F+f)}=\frac{y_e-\rho}{F+f}.$$

The angular radius Θ_2 of the extreme field is the angle made by O_2D_1 with the axis.

Hence $$\Theta_2=\frac{y_eF+y_of}{F(F+f)}=\frac{y_e+\rho}{F+f}.$$

If we put $y_e=mf$, ($m=1/4$ to $1/6$ for eye-glasses), $y_o=nF$, ($n=1/20$ to $1/30$ for object-glasses), and if M be the (numerical) magnifying power, then the value of Θ_2 in circular measure is $(m+n)/(M+1)$. Hence all telescopes of large magnifying power have a very small field, and are therefore provided with a *finder*, which is a smaller telescope of less power rigidly attached to the other, so that their axes are parallel.

The rays from stars, visible by pencils that fill the object-glass, will cross the common focal plane at F_2 within a circle of radius $(y_eF-y_of)/(F+f)$; and a stop of this size is usually put in telescopes to cut off the rest of the field of view, which is not so bright as that seen by full pencils, and is known as the ragged edge.

In telescopes used for astronomical measurements a micrometer is also fixed in the focal plane of the object-glass, so that the star images may coincide with it. This must be viewed together with those images through the eye-piece; so that it is impossible to use a micrometer with those eye-pieces, as Huyghens', in which the first focus falls within the eye-piece.

If the aperture of the object-glass is large and the magnifying power not very great, it may happen that the eye-ring is larger than the pupil of the eye. In that case the angular radius of the field of view seen by pencils that ultimately fill the pupil of the eye of radius p, but do not initially fill the object-glass, is $(y_e-p)/(F+f)$, and that of the extreme field is $(y_e+p)/(F+f)$ the eye being still placed at the eye-ring.

103. Readjustment for finite distance of image.

Each observer on looking through a telescope will arrange it so that the final image comes at the most convenient distance from his eye. This distance is spoken of as the "distance of distinct vision"; a misleading term, as vision is distinct at all distances within the range of accommodation of the eye.

If we suppose that the eye is passive, as seems probable from the fact that there are not objects at different distances in view to suggest accommodation, this "distance" will range from a small limit for a myopic eye through infinity for a normal eye to a negative limit for a hypermetropic eye.

Let it be denoted by Δ; then the final image of a distant object is to be at distance Δ from the eye, and the eye-piece will be moved from its position in normal adjustment through a distance x depending on the position of the eye.

(i) Let the eye be placed at the new eye-ring; the object-glass is now $F-x$ in front of f_1, and the new eye-ring is therefore $f^2/(F-x)$ behind f_2.

The first image of a star is formed at F_2, at distance x behind f_1, and the final image is therefore f^2/x in front of f_2.

Hence $$f^2/x+f^2/(F-x)=\Delta.$$

Therefore $$x^2-xF+f^2F/\Delta=0,$$

and $$x=\tfrac{1}{2}\{F-(F^2-4Ff^2/\Delta)^{\frac{1}{2}}\}.$$

To determine the magnifying power. Let σ be the image of a star formed on the second focal plane of the object-glass, and σ' the image of σ formed by

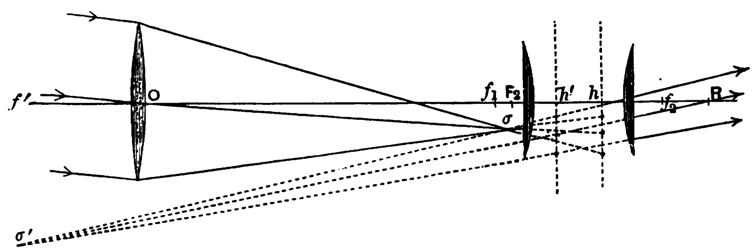


Fig. 58.

the eye-piece. The principal ray $O\sigma$ emerges as $\sigma'R$, passing through the first nodal point of the eye at R, and the magnifying power is the ratio of the angles $\sigma'RO$ and σOR, since the incident rays were all parallel to $O\sigma$.

Its value is therefore (algebraically)

$$=-\frac{\sigma'f'}{\Delta}\Big/\frac{\sigma F_2}{F}=-\frac{F}{\Delta}\frac{f}{x}=-\frac{F+(F^2-4Ff^2/\Delta)^{\frac{1}{2}}}{2f};$$

since $\sigma'f'/\sigma F_2=$ the linear magnification of σF_2 produced by the eye-piece $=f/x$.

(ii) Let the eye remain at the eye-ring for normal adjustment, in contact with the eye-stop.

This is f^2/F behind f_2, and hence the equation for x in this case is

$$f^2/x + f^2/F = \Delta.$$

Hence . $x = f^2/(\Delta - f^2/F)$.

In this case the principal ray, which enters the first nodal point E of the eye, and on which the final image σ' must be considered to lie, will be found to correspond to an incident ray crossing the axis at distance f^2/Δ in front of the object-glass; but the angle made with the axis by all incident rays from the star is the same and equal to $\sigma O F_2$.

Hence the magnifying power is still the ratio of the angles $\sigma' E O$ and $\sigma O E$, and is equal as before to $-\dfrac{\sigma' f'}{\Delta} \Big/ \dfrac{\sigma F_2}{F}$ or $-\dfrac{F}{\Delta}\dfrac{f}{x}$, *i.e.* in this case to $-\dfrac{F}{f}\left(1 - \dfrac{f^2}{F\Delta}\right)$.

It is well known that a very small displacement of the eye-piece destroys the clearness of the image. This is due to the fact that such a displacement involves a very much greater displacement of the image, which may carry it beyond the range of accommodation. For if x become x', then the image is at distance Δ', where $\Delta' - \Delta = \dfrac{f^2}{xx'}(x - x')$. As Δ and Δ' are certainly much greater than f, x and x' are much less than f, and the displacement of the image will be very large.

104. Eye-pieces.

An eye-piece, consisting as a rule of two convergent lenses, is used in the different forms of telescopes, and in the microscope, in place of a single eye-glass. By this means the achromatism of the image is better secured, and a larger and flatter field of view can also be obtained.

We shall see (Chap. VII.) that the condition of achromatism, as far as regards the linear magnification and not the positions of the images, is that the distance between the two lenses be the arithmetic mean of their focal lengths.

The angular radius of the field of view of an astronomical telescope fitted with an eye-piece may be obtained by finding the images of the lenses in the previous part of the telescope, and according to the apertures of the object-glass and the lenses the various cases of Art. 98 may arise.

Generally the lens of the eye-piece nearer the object-glass governs the field of view; this lens is called the *field-glass*.

The positions of the principal foci of the eye-piece, and its focal length may be found in any case by Cotes' formulæ; these points will take the place of the foci of a single lens in the

adjustment of the astronomical telescope, and its focal length will be used in determining the magnifying power.

In *Huyghens' eye-piece* the focal length of the field-glass is three times that of the eye-glass, and the distance between the two lenses is twice the latter focal length.

The lenses used are usually plano-convex, having both their plane faces to the eye. With lenses of this shape, which are easy to work, the above ratio of the focal lengths (or rather a slightly smaller ratio 2·6) minimises the distortion and the curvature of the image formed by the eye-piece of an object at its first focus.

Neither these curvatures of the lenses nor the ratio of the focal lengths are necessarily suited to give a flat image of objects at other distances from the eye-piece; but the analysis necessary to determine the curvatures is very complicated (see below Chap. XIV.), while only certain curves are practically attainable in shaping the lenses.

If f be the (numerical) focal length of the eye-glass A_2, and $3f$ that of the field-glass A_1 (Fig. 59), then $A_1A_2 = 2f$ and to determine the first focus of the system, we have by Cotes' formula I.,

$$0 = 1 - u/3f - (u + 2f)/f + u \,.\, 2f/3f^2\,;$$

in which the coefficient of u is the power of the eye-piece.

The equivalent focal length of the eye-piece is therefore $\frac{3}{2}f$ numerically.

Also $u = -\frac{3}{2}f$; the first focus is between A_1 and A_2; and this eye-piece cannot be used for a telescope containing a micrometer. For the rays from the object-glass to the image at f_1 are intercepted by the field-glass, which forms a real, but slightly distorted, image at the focus of the eye-glass, midway between A_1 and A_2. A micrometer would naturally be placed there, that its image may apparently coincide with the final image, but then the virtual image would be seen by rays that are refracted at both lenses of the eye-piece, and suffer counterbalancing distortions, while the micrometer would be seen by rays passing through the eye-glass only, and would be distorted differently.

The second principal focus is at distance $\frac{1}{2}f$ behind the eye-glass; the first and second unit points are at distance f respectively behind and in front of it. The (hypothetical) path of a pencil converging initially to a virtual focus on the focal plane through f_1 can be easily drawn.

In the figure the three incident rays, converging to the point σ on the first focal plane, meet when produced the first unit plane in the points p_1, p_2, p_3; the corresponding emergent rays are parallel to σh, and meet the second unit plane in the points p_1', p_2', p_3' respectively if produced backwards.

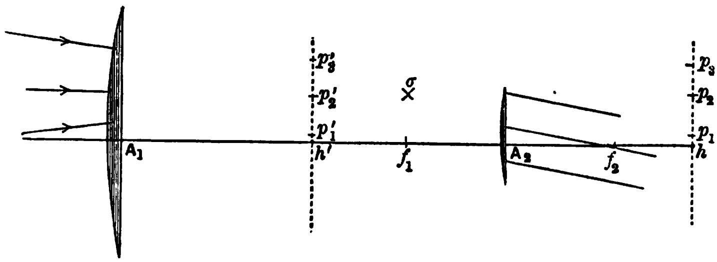


Fig. 59.

In *Ramsden's eye-piece* the lenses are of equal focal lengths. The condition of achromatism would therefore require that their distance apart be also the focal length; but then the first focus of the system would fall at the field-glass itself, and no micrometer could be used, while the field-glass would be visible with the image through the eye-glass. The distance apart is therefore made slightly less than the focal length; and as a rule is taken to be two-thirds the focal length. The lenses are plano-convex, having their plane faces outwards; and this arrangement with the given distance between the lenses produces a very flat field of view, comparatively free from indistinctness and distortion.

If f be the (numerical) focal length of each lens, that of the combination is $\frac{3}{4}f$; the principal foci are at a distance $\frac{1}{4}f$ in front of the field-glass, and behind the eye-glass respectively, and the unit points fall between the lenses at a distance $\frac{1}{2}f$ from the field-glass and eye-glass respectively.

The positions of these points are given in Fig. 58.

The *erecting eye-piece* is used in telescopes for viewing terrestrial objects. Both Huyghens' and Ramsden's eye-pieces have convergent powers, and the image appears inverted; it is necessary therefore to form an eye-piece with divergent power. For this purpose two convergent eye-pieces, placed so that the second focus of the first precedes the first focus of the second, are sufficient. If f_e, f_e' be the focal lengths of the two eye-pieces and c the distance

between the foci mentioned, the focal length of the combination is $f_e f_e'/c$.

The eye-piece may be made achromatic by choosing the intervals between the lenses, and distortion and curvature of the image removed by choosing the curvatures properly; but as the telescope will not be used for magnifying powers exceeding 20 or so, these errors are of less importance, and are left uncorrected.

The four lenses are fitted in a tube, and retain their relative positions when any readjustment of the telescope is made.

105. Galileo's Telescope.

This instrument consists of a convergent object-glass and a divergent eye-glass. Hence in normal adjustment the common focus of the two lenses falls behind the eye-glass, and their distance apart is the difference of their focal lengths.

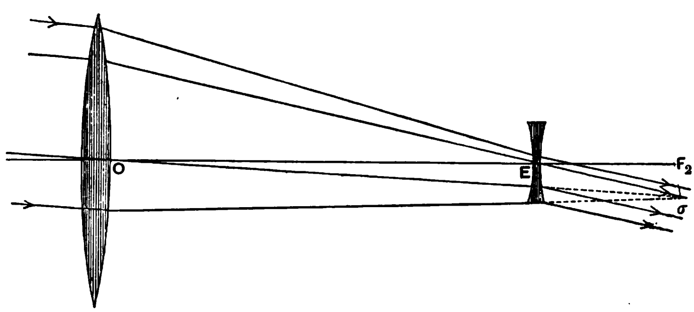


Fig. 60.

A pencil of parallel rays after refraction through the object-glass converges towards a point σ on the common focal plane, but is intercepted by the eye-glass, and leaves it as a pencil of rays parallel to $E\sigma$. Since the inclination of the axis of the pencil to the axis of the telescope remains of the same nature, and is increased, the image of any distant object is erect and magnified.

The eye-ring falls within the telescope, so that the eye must be placed at the eye-glass; and since the rays diverge from the axis after leaving that lens it is useless to construct it of aperture larger than that of the pupil of the eye.

The magnifying power M is, in the case of normal adjustment, equal to F/f, as in the astronomical telescope.

Since this instrument gives an erect image and its length need not be very great for low magnifying powers, it is well adapted for viewing terrestrial objects; and the opera-glass consists of a pair placed with their axes parallel. The eye-glass will usually consist of an achromatised doublet, the lenses being in contact, since the incidence on them is central.

106. Field of View.

The field of view is determined as before by the entrance-

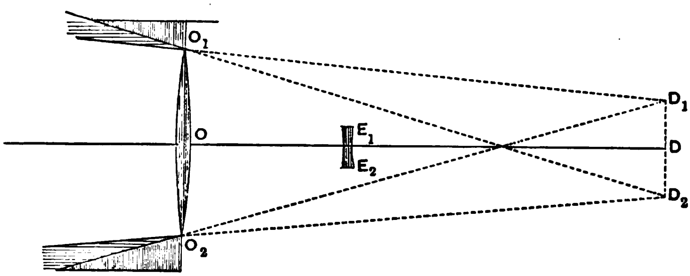


Fig. 61.

pupil D_1D_2, which is now virtual, so that $OD = F^2/f - F$. Its radius is My_e, and according as My_e is greater or less than y_o, the cases iii. and iv. of Art. 98 may arise. If the entrance pupil be less than the object-glass, the eye-glass will, in normal adjustment, be less than the eye-ring; and the pencils that fill the eye-glass will consist of rays incident initially on part only of the object-glass (Fig. 61).

This is the more general case, and it is clear that, if D_1O_1 and D_2O_2 be produced beyond the object-glass, the lines joining any point within the region so defined to D_1D_2 will mark the limits of an incident pencil of rays, which after refraction at the object-glass will completely fill the pupil E_1E_2. Hence the angular radius Θ_1' of the part of the field, which is seen by pencils filling the pupil, and therefore appears of uniform brightness, is given by

$$\begin{aligned}\Theta_1' &= \left(y_o - y_e\frac{F}{f}\right) \Big/ \left(\frac{F^2}{f} - F\right)\\ &= (y_o f - y_e F)/F(F-f)\\ &= (\rho - y_e)/(F-f),\end{aligned}$$

where ρ is the radius of the eye-ring.

The angular radius Θ_2' of the extreme field is the angle between D_2O_1 and the axis; and we have

$$\Theta_2' = (y_o f + y_e F)/F(F-f) = (\rho + y_e)/(F-f).$$

The pencils from points at angular distance from the axis between Θ_1' and Θ_2' will not fill the pupil, and therefore the edge of the field will appear less bright than the centre; but as the first image is formed in the common focal plane and is therefore virtual, no stop can be placed to cut off this ragged edge.

In the case where the entrance-pupil is greater than the object-glass, the pupil of the eye will also be greater than

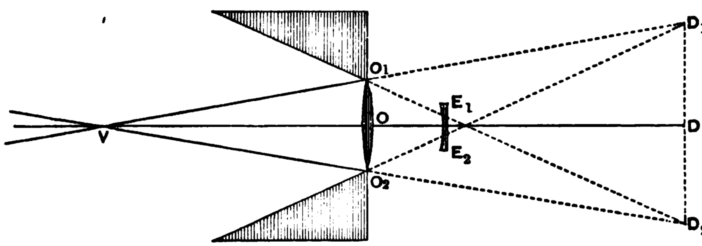


Fig. 62.

the eye-ring, and therefore greater than the pencil of rays from a distant object as they emerge from the eye-glass (Fig. 62).

It is then plain that objects lying in the semi-infinite cone beyond V will be seen by pencils that fill the object-glass but not the pupil. The angular radius Θ_1 of the field is given by

$$\Theta_1 = (y_e F - y_o f)/F(F-f)$$
$$= (y_e - \rho)/(F-f).$$

The extreme field as before is found by joining D_1O_2 and D_2O_1, and we have

$$\Theta_2 = (y_e F + y_o f)/F(F-f) = (y_e + \rho)/(F-f).$$

107. Adjustment for finite distance of image.

The readjustment for a distance of distinct vision Δ is simpler in this case than in the astronomical telescope, as the eye is at the eye-glass.

Let the eye-glass be pushed in a distance x; the image σ of a distant object is formed on the focal plane of the object-glass at distance x behind f_1, and the final image σ' is at distance f^2/x in front of f_2.

Hence $$f^2/x = \Delta - f.$$

The magnifying power is, as before, the ratio of the angles $\sigma' EO$ and σOE, and we have

$$M = \frac{F}{\Delta}\frac{f}{x} = \frac{F}{f}\left(1 - \frac{f}{\Delta}\right).$$

Plainly the rays are incident excentrically on the object-glass; and they are divergent after leaving the eye-glass.

As this instrument is used for terrestrial objects, we may calculate the readjustment x' from the positions of the lenses when in normal adjustment for a very distant object to their positions when an object at finite distance is seen as follows.

Let an object of linear dimensions l be at a finite distance u in front of the object-glass; its distance from F_1 is $u - F$, and the first image, of linear dimensions l', is at distance $F^2/(u-F)$ behind F_2. This is $x' + F^2/(u-F)$ behind f_1, and the final image, of linear dimensions l'', is at distance $f^2/\{x' + F^2/(u-F)\}$ in front of f_2. But the eye being at the eye-glass, this must be equal to $\Delta - f$.

Hence
$$\frac{f^2}{\Delta - f} - \frac{F^2}{u - F} = x'.$$

The magnifying power, *i.e.*, in this case the angular magnification for an object at this finite distance, is

$$\frac{l''}{\Delta} \bigg/ \frac{l}{u + F - f - x'};$$

which is equal to

$$\frac{u + F - f - x'}{\Delta}\frac{l''}{l'}\frac{l'}{l} = \frac{u + F - f - x'}{\Delta} \cdot \frac{\Delta - f}{f} \cdot \frac{F}{u - F}$$
$$= \frac{F}{f}\left\{\frac{u^2}{(u-F)^2}\left(1 - \frac{f}{\Delta}\right) - \frac{f}{u - F}\right\}.$$

108. Newton's Telescope.

The first reflecting telescope was constructed by Newton in 1668; he was led to invent this form owing to his belief that achromatic refractors were impossible.

In this telescope the object-glass is replaced by a concave mirror; the rays reflected at this are intercepted by a plane mirror placed on its axis between the vertex and focus, so that the reflection in the plane mirror of the image that would be formed at the focus is visible by an eye-piece placed at the side of the tube.

The plane mirror is inclined at an angle of 45° to the axis, so that the image which it forms is parallel to the axis.

An incident pencil of parallel rays will be reflected by the

concave mirror towards a point σ on its focal plane (Fig. 63); these rays are intercepted by the plane mirror and converge to the image σ', which in normal adjustment lies in the first focal plane of the eye-glass; and they finally emerge from the instrument as a pencil of rays parallel to $\sigma'E$, the principal ray meeting AE in R, the eye-ring.

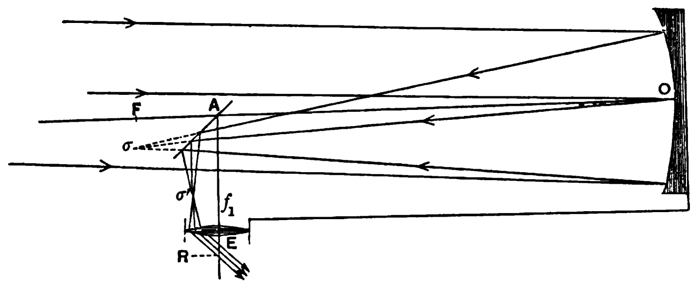


Fig. 63.

The small mirror forms an image of the concave mirror at a distance along EA produced equal to AO. The appearance of the field and the necessary adjustments of the instrument are exactly the same as if this image were the object-glass of an astronomical telescope. If F be the focal length of the mirror, f that of the eye-glass, the condition of normal adjustment is $AO + AE = F + f$, and the magnifying power is F/f.

The alteration in the magnifying power, and the readjustment of the eye-glass necessary when the final image of a star is at distance Δ from the eye, can be determined exactly as in the case of the astronomical telescope (Art. 103).

So also the field of view visible by pencils that fill the concave mirror is of angular radius $(y_e F - y_0 f)/F(F+f)$ in normal adjustment.

In some instruments an isosceles right-angled glass prism takes the place of the plane mirror; the rays are then bent through a right angle by internal reflection, and by this method less light is lost than by reflection at a metallic surface.

109. Shape and dimensions of the plane mirror.

Since the plane mirror must necessarily intercept part of the incident light, and that part contains the rays which would be reflected near the vertex of the concave mirror and suffer least

from aberration, it is essential to make it as small as possible. At the same time it must be large enough to intercept and reflect all rays of pencils that will fall within the eye-glass.

In Fig. 64 let E_1EE_2 be the image of the eye-glass formed by the plane mirror; let D_1DD_2 (not included in the figure) be the image of E_1EE_2 in the concave mirror. The angular radius Θ of the distant field of view is, as before, in normal adjustment, equal to

$$(D_2D - O_1O)/OD,$$

or

$$(y_eF - y_of)/F(F+f).$$

Consider the pencil of parallel rays from a star in the direction of D_1; they will be reflected towards a point e_1 on the focal plane through F. The extreme ray D_1O_2 will be reflected as $O_2e_1E_1$; the parallel ray incident at O_1 will be reflected towards e_1 and cut the axis in a point V between F and E, and E_1E_2 at a point within E_1E_2. This full incident pencil will therefore emerge through the corresponding part of the eye-glass.

Similarly all the incident pencils which make the extreme angle Θ with the axis are reflected as cones of rays having their vertices on the focal plane at distance Fe_1 from F; and pencils making less than this angle are reflected as cones with their vertices at points on the focal plane within this distance from F.

Plainly all the reflected rays will be included within the cone, vertex V, and base the object-mirror, and within no smaller space. The necessary shape of the plane mirror is therefore the *elliptic* section of this cone by a plane at A making an angle 45° with its axis.

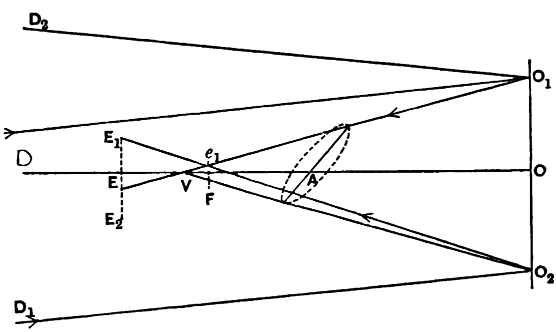


Fig. 64.

As the semi-vertical angle α of this cone has been assumed a small quantity whose square may be neglected, it is easy to see that to this order of approximation A is the centre of the ellipse, the semi-major axis $= \sqrt{2}\, VA \,.\, \alpha$, and the semi-minor axis = radius of the circular section through A, $= VA \,.\, \alpha$.

The amount of incident light stopped by the plane mirror is approximately proportional to the area of its projection on a plane perpendicular to the axis, *i.e.* to $\pi VA^2\alpha^2 = \pi y_0^2\,(VA/VO)^2$.

The fraction stopped is therefore $(VA/VO)^2$.

Also if b denote the distance FA, which in normal adjustment is equal to the distance from A to the first focus of the eye-glass, we have

$$\Theta + \alpha = y_0/F;\quad VO = y_0/\alpha \text{ and } VA = y_0/\alpha - F + b.$$

Hence
$$VA/VO = 1 - \frac{F-b}{y_0}\left(\frac{y_0}{F} - \Theta\right)$$
$$= \frac{b}{F} + \frac{F}{y_0}\Theta - \frac{b}{y_0}\Theta.$$

Practically b does not differ much from y_0, and the first two terms of this expression are small quantities of about the same magnitude, while the last term is small compared with either.

Not much above one per cent. of the light will as a rule be intercepted.

110. Herschel's Telescope.

A concave mirror will form a real image in its focal plane of a distant object. A micrometer may be placed there and viewed with the image through an eye-piece. Since the observer's head cannot come directly in the axis of the telescope, Herschel placed the mirror obliquely to the axis, so that the image was formed near the edge of the tube.

If we neglect the obliquity of the axis of the mirror OM to the axis of the tube OA, then the image of a star S is formed at σ in the focal plane, and viewed through an eye-piece as in the astronomical telescope.

The magnifying power is $-F/f$; for the angle SOA between the rays from two distant objects is equal to the angle σOF

between the reflected rays, and the angle between the emergent rays is σEF.

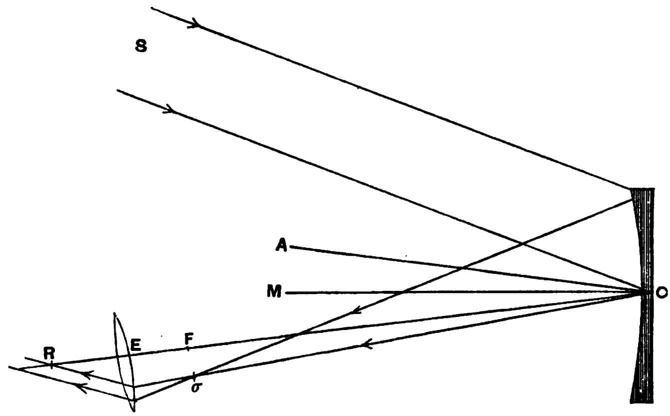


Fig. 65.

The defect of this telescope is the obliquity of the pencils, producing distortion and indistinctness in the images. These faults increase rapidly with the inclination of the rays to the axis of the mirror. In Herschel's own instrument the aperture was 4 feet, and the focal length 40 feet; which would indicate an obliquity of about 3°. The great advantage is the decrease in the number of reflections or refractions, and the consequent gain of light; the aperture also of the mirror can be much greater than that of any lens procurable.

111. Gregory's and Cassegrain's Telescopes.

The first plan of a reflecting telescope was devised by James Gregory in 1663, but it was not actually carried out till after Newton's telescope had been constructed. Gregory's telescope is described as consisting of a large concave mirror having an aperture at its vertex and a smaller mirror, also concave, placed on its axis. The focus of this mirror is slightly beyond that of the large mirror, so that any pencil of incident parallel rays converging after reflection to a point on the focal plane of the object-mirror is reflected to a point lying near the aperture. They then fall on an eye-glass or eye-piece, and in normal adjustment emerge as a pencil of parallel rays. Readjustment for a finite distance of distinct vision is obtained by moving the smaller

mirror. Since each reflection at the concave mirrors inverts the image, objects are seen erect with this instrument.

Cassegrain's telescope, invented in 1672, differs from Gregory's only in having the second mirror convex to the light instead of concave. It therefore gives an inverted image of any distant object, but it is superior to Gregory's in two points: first, the spherical aberrations of the two mirrors tend to correct instead of reinforcing each other, thus promoting good definition of the image; secondly, the necessary radius of aperture of the convex mirror is less, so that the proportion of light stopped is less in this instrument than in Gregory's telescope.

For these reasons Cassegrain's telescope is now always preferred to Gregory's; the Melbourne equatorial, of 4 feet aperture and $30\frac{1}{2}$ feet focal length, is of this form.

112. Adjustment and magnifying power.

In Fig. 66, O is the vertex and F the focus of the object-

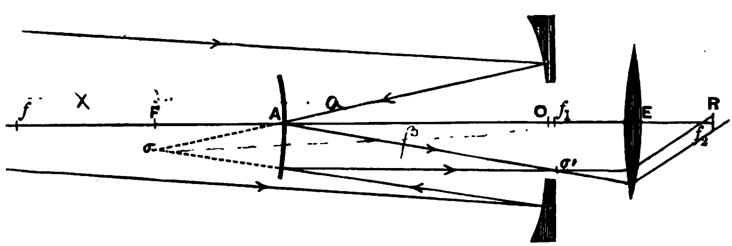


Fig. 66.

mirror; A, f, corresponding points for the convex mirror; f_1 and f_2 are the foci of the eye-glass E, and R is the eye-ring.

The breadth and the focal length of the object-mirror are necessarily much diminished in the figure in comparison with those of the convex mirror.

Let F, f_m, f_e be the (numerical) focal lengths of the object-mirror, the convex mirror and the eye-glass (or eye-piece) respectively. Also let a denote the distance Ff_1; this is a constant of the instrument, and as a rule differs very slightly from F.

In normal adjustment the points F and f_1 are conjugate to each other in the convex mirror; hence, if x denote the distance fF between the foci of the two mirrors, the condition of normal adjustment is

$$x(x+a)=f_m^2.$$

The positive root, $\frac{1}{2}\{(a^2+4f_m^2)^{\frac{1}{2}}-a\}$, must be taken, and as a first approximation $x=f_m^2/a$, a being large compared with f_m.

A pencil of parallel rays incident on the telescope will first be reflected towards the focus σ on the focal plane of the object-mirror. Before reaching σ they are intercepted by the convex mirror and brought to a focus at σ', which, in normal adjustment, is in the focal plane of the eye-glass; the rays then emerge from the eye-glass parallel to $\sigma'E$.

To find the magnifying power: the angle made by the incident rays with the axis of the telescope is equal to that subtended by σF at the centre or at the vertex of the object-mirror.

The magnifying power is therefore equal to the ratio of the angles $\sigma'Ef_1$ and σOF; *i.e.* it is equal to

$$\frac{\sigma'f_1}{f_e}\bigg/\frac{\sigma F}{F}=\frac{F}{f_e}\frac{f_m}{x}=\frac{F(a+x)}{f_ef_m};$$

by the formula for linear magnification at a mirror.

This is its numerical value; a distant object is however inverted and perverted, as in the astronomical telescope. For the reasons given above an approximate value of the magnifying power is F^2/f_ef_m.

113. Eye-ring and field of view.

The distance of the object-mirror from f is $x+F$; its first image is therefore at distance $f_m^2/(x+F)$ from f. This is at distance $x+a-f_m^2/(x+F)$ in front of f_1, which is, *in normal adjustment*, equal to $f_m^2/x-f_m^2/(x+F)$, or $f_m^2F/x(x+F)$. The eye-ring is therefore at a distance behind f_2 equal to $f_e^2x(x+F)/f_m^2F$, which is equal to $\frac{f_e^2}{F}\left\{1+\frac{(F-a)x}{f_m^2}\right\}$, and differs inappreciably from f_e^2/F.

The field of view may be governed by the apertures of the mirrors as well as by that of the eye-glass. However, the convex mirror should be taken of sufficient aperture to intercept all the reflected rays of the full field of view; and then the full field is

completely determined by the object-glass and the entrance-pupil. The entrance-pupil lies in front of the instrument; for in the figure E is further from f than f_1, and therefore the first image E' of the eye-glass, which is formed by the convex mirror, lies between f and F. The entrance-pupil D, which is the image of E' in the object-mirror, is therefore to the left of F or in front of the telescope.

We may then find the field of view as follows. Let y_o, y_e be the radii of aperture of the object-mirror and the eye-glass respectively; let M be the numerical magnifying power. Let d denote the distance of the eye-ring behind the eye-glass; then

$$d = f_e + \frac{f_e^2}{F}\left\{1 + \frac{(F-a)\,x}{f_m^2}\right\}.$$

By the properties of instruments in normal adjustment the radius of the eye-ring is y_0/M, which we suppose less than y_e. The angle therefore made by the emergent ray E_1R_1 (Fig. 67) with the axis is $\left(y_e - \frac{y_0}{M}\right) \Big/ d$. Hence the angle made by the corresponding incident ray D_1O_1 with the axis is $\left(y_e - \frac{y_0}{M}\right) \Big/ Md$. This is therefore the angular radius Θ of the field of view seen by pencils that fill the object-mirror.

114. Aperture of the convex mirror.

As in the case of Newton's telescope, the necessary radius of aperture of the convex mirror may be obtained, with the in-

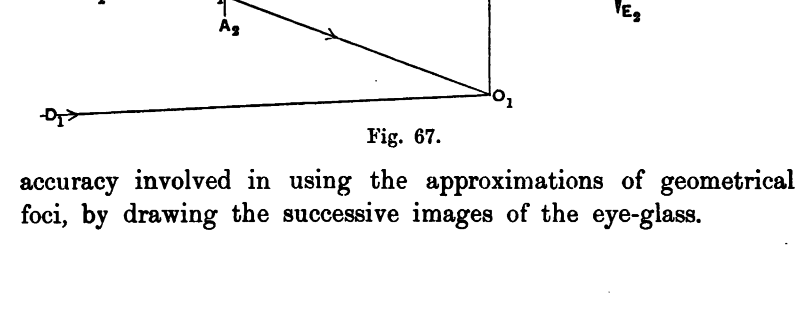


Fig. 67.

accuracy involved in using the approximations of geometrical foci, by drawing the successive images of the eye-glass.

Let the instrument be in normal adjustment, and in Fig. 67 let E_1E_2 be the eye-glass, $E_1'E_2'$ the image of E_1E_2 in the convex mirror. Then $E_1'E_2'$ will be erect, and within fF.

The entrance-pupil D_1D_2 (not included in the figure), is in front of the object-mirror and inverted. The boundaries of the field of view seen by full pencils are obtained by joining O_1D_1 and O_2D_2.

A pencil of parallel rays from a very distant point in the direction D_1O_1 will be reflected towards the point e_1' on the focal plane through F; they are next reflected by the convex mirror to the point e_1 on the focal plane through f_1, and emerge from the eye-glass as a pencil of parallel rays. If we consider the extreme rays of this pencil, we see that the lower ray D_1O_1 in the axial plane is reflected through e_1' to E_1' (in this way e_1' is determined), and in its path cuts the plane of the convex mirror in B_1, and the plane of the object-mirror in Q. The parallel upper ray in the axial plane, incident at O_2, cuts these planes after reflection in A_1 and P respectively. In all cases $AA_1 > AB_1$, since e_1' is above the axis; and $OP > OQ$, if, as is necessary for using a micrometer, f_1 lie behind O.

The radius of aperture of the convex mirror necessary that all rays from stars in the field of view may be intercepted by it and reflected, is therefore AA_1; the necessary radius of the hole in the object-mirror, to allow these rays to pass, is OP.

To find AA_1 $(=y_m)$, let α be the inclination of the ray O_2A_1 to the axis; then with the notation of the previous article,

$$\alpha + \Theta = y_o/F.$$

Also $$y_o - y_m = AO \, . \, \alpha = (F + x - f_m)\,\alpha.$$

Hence $$y_m = y_o - (F + x - f_m)\,\alpha$$
$$= (f_m - x)\, y_o/F + (F + x - f_m)\,\Theta.$$

Again, if α' be the angle of divergence of the ray A_1P, we have $\alpha + \alpha' = y_m/f_m$; and the convex mirror will be larger than the aperture, so that no rays pass through directly, if α' be negative. The condition for this is that

$$xy_o/F > (F + x)\,\Theta.$$

The radius of the aperture OP is then given by the equation

$$y_a = y_m - (\alpha - y_m/f_m)(F - f_m + x)$$
$$= (F + x - f_m)(F + x + f_m)\,\Theta/f_m + y_0(f_m^2 - Fx - x^2)/Ff_m,$$

on substitution for y_m and α.

In Gregory's telescope, the values given above for the magnifying power and for the field of view are unchanged.

But in calculating the necessary radius of aperture of the smaller mirror, it will be found that the entrance-pupil D_1D_2 is erect, and that it is the extreme ray bounding the field of view which after reflection cuts the small mirror at the greater distance from the axis, and not, as in Cassegrain's telescope, the ray parallel to this one. Also since OA is now $F+x+f_m$, we find for the radius of aperture

$$y_m = (f_m + x)\,y_0/F + (F + x + f_m)\,\Theta,$$

a value larger than that required in the other construction.

115. As an illustration of the formulæ of Art. 87, we may at once write down the values of y_m for the convex mirror and of y_a for the aperture in the concave mirror. The successive refractive indices must be taken as $+1$, -1, $+1$, the power of the concave mirror as $(-1-1)/R$ or $-1/F$, the power of the convex mirror in Cassegrain's telescope as $(1--1)/r$ or $1/f_m$, and the reduced distance $\overline{OA}$ as $-(F-f_m+x)/(-1)$.

Hence y_m/Θ = apparent distance of A from the point where the incident ray cuts the axis,

$$= u + (F - f_m + x) - u\,(F - f_m + x)/F.$$

For the incident upper ray (Fig. 67), $u\Theta = y_0$, and hence

$$AA_1 = y_m = (f_m - x)\,y_0/F + (F - f_m + x)\,\Theta.$$

For the incident lower ray, $u\Theta = -y_0$ and

$$AB_1 = y'_m = -[(f_m - x)\,y_0/F - (F - f_m + x)\,\Theta].$$

Again, the heights at which these rays cross the large mirror are given by the apparent distance of O, as seen by two reflections, from their origins on the axis.

For the first ray,

$$y_a/\Theta = u + \overline{OA} + AO - u(\overline{OA} + AO)/F + (u + \overline{OA})AO/f_m - u\,.\,\overline{OA}\,.\,AO/Ff_m,$$
$$= u + 2(F - f_m + x) - u\,.\,2\,(F - f_m + x)/F + (u + F - f_m + x)\,(F - f_m + x)/f_m$$
$$- u\,(F - f_m + x)^2/Ff_m,$$

i.e. $$OP = y_a = (F - f_m + x)\,(F + f_m + x)\,\Theta/f_m + y_0\,(f_m^2 - Fx - x^2)/Ff_m$$
$$= \{F^2 + x\,(2F - a)\}\,\Theta/f_m + y_0 x\,(a - F)/Ff_m,$$

on substituting from the equation $x\,(x + a) = f_m^2$.

And similarly $OQ = \{F^2 + x\,(2F - a)\}\,\Theta/f_m - y_0 x\,(a - F)/Ff_m$.

116. Readjustment for finite distance of distinct vision.

Let the instrument be so adjusted that the final image of a star is formed at distance Δ from the eye. (i) As the eye is usually placed in contact with the end of the tube, we may suppose it to occupy a fixed position whatever be the adjustment of the telescope. Let δ be its distance behind the second focus, f_2, of the eye-piece. Then $\Delta - \delta$ is the distance of the final image in front of f_2; denote this by Δ'. Let y be now the distance between the foci f and F of the mirrors; we suppose that the readjustment is made by moving the smaller mirror.

Let the first image of a star, lying off the axis of the telescope, be formed at σ on the focal plane of the large mirror at distance l from the axis, and let the second image, formed by the convex mirror, be at σ' at distance l' from the axis. Let the final image be at distance l'' from the axis. Then since the focal plane on which σ lies is at distance y from f, the distance of σ' from that point is f_m^2/y, and since the distance of f_1 from f is now $y+a$, the distance of σ' *behind* f_1 is $f_m^2/y-(y+a)$. This must be equal to f_e^2/Δ'.

The distance between the foci of the mirrors in this adjustment of the telescope is therefore given by the positive root of the equation

$$\frac{f_m^2}{y}-(y+a)=\frac{f_e^2}{\Delta'}.$$

The corresponding distance in normal adjustment is given by the equation

$$f_m^2=x(x+a).$$

Hence the distance $(x-y)$ which the small mirror must be moved is equal to $yf_e^2/\Delta'(x+y+a)$. As Δ' is certainly large compared with f_e, and as x and y are small compared with a, the value $f_m^2f_e^2/a^2\Delta'$ may be taken as a first approximation.

The magnifying power in this arrangement of the telescope is $\dfrac{l''}{\Delta}\Big/\dfrac{l}{F}$, since the principal emergent ray, which enters the eye, makes the angle l''/Δ with the axis, and all the incident rays make the angle l/F with the axis.

Hence $$M=\frac{l''}{l'}\cdot\frac{l'}{l}\cdot\frac{F}{\Delta}=\frac{\Delta'}{f_e}\cdot\frac{f_m}{y}\cdot\frac{F}{\Delta}=\frac{Ff_m}{yf_e}\left(1-\frac{\delta}{\Delta}\right).$$

(ii) If, however, we suppose the eye to be placed at the new eye-ring, the position of which will depend on the adjustment of the telescope, then the distance y between the foci is given by the equation

$$\Delta=f_e^2\Big/\left\{\frac{f_m^2}{y}-(y+a)\right\}+f_e^2\Big/\left\{y+a-\frac{f_m^2}{y+F}\right\};$$

in which the second term is the distance of the new eye-ring behind f_2.

The actual method of adjustment used in the Melbourne instrument is quite different from that stated above. For micrometer work a field-glass common to all the eye-pieces is placed at a proper distance before the wires, and the eye-pieces are single lenses placed behind them. The small mirror is then fixed in the position that brings the third image in the plane of the wires, and focusing for different distances of distinct vision is effected by moving the eye-lens.

MICROSCOPES.

117. If we wish to distinguish the details of a small object we naturally place it at the nearest distance of distinct vision. The power of the eye to estimate detail may then be measured by the size of the retinal image of unit length of the object, and therefore by the angle which unit length of the object subtends at the eye. This will be $1/\Delta'$, where Δ' is the least distance of distinct vision; and hence for viewing small objects a myopic eye has a superiority over the emmetropic eye.

If the object be brought still nearer to the eye, the divergence of the rays becomes too great for the power of accommodation of the eye; and it is clear that the use of a convergent lens will diminish the divergence, and remove the image to a possible distance of distinct vision.

Let then a small object PQ be placed between the first focus

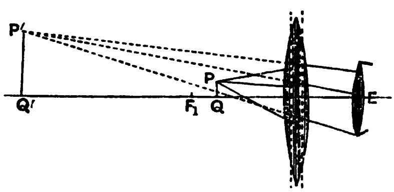


Fig. 68.

F_1 of a convergent lens and the lens itself. An erect, virtual and magnified image $P'Q'$ will be formed, and may be viewed by the eye E behind the lens as if it were a real object.

Let the distance EQ' of the image from the eye be equal to Δ; then the *amplifying power of the lens, i.e. the angle under which unit length of the object is seen,* is $P'Q'/\Delta$, where $P'Q'/(1)$ is the linear magnification, which is equal to F_2Q'/f, *i.e.* to $(F_2E+\Delta)/f$.

Hence we obtain the expression given by Abbe* for the amplifying power, namely,

$$\frac{1}{f}+\frac{1}{f}\frac{F_2E}{\Delta}.$$

* Abbe, *Journal Royal Microscopical Society*, 1884, p. 350.

When the eye lies between the lens and its second focus, this expression is greatest when Δ is least, and when the eye is as close to the lens as possible; also the field of view has then its greatest possible value.

But if the eye lie behind the second focus, then the amplifying power is greatest when Δ is greatest, and an emmetropic eye, for which Δ can be infinite, will obtain the greatest amplifying power ($1/f$) when the object is placed at the first focus. The rays from the various points of the object will then emerge as parallel pencils in different directions; and it is at once obvious from a figure that we have practically reduced the distance of distinct vision to the focal length of the lens without making any demand on the accommodation of the eye.

118. The amplifying power attainable by a single lens is not very great for several reasons. A short focal length allows of very scanty frontal distance (*i.e.* the distance of the object from the lens), and is therefore inconvenient for working. The defects of aberration and distortion for an object so near the lens are comparatively large; and the image formed by a single lens is also largely affected by chromatism. For these reasons the single lens is replaced by several lenses forming a *microscope*. The formula for the *amplifying power** remains as before, f being the focal length and F_2 the second focus of the entire system.

The compound microscope is regarded as made up of two parts, the objective and the ocular. The objective consists of several lenses, and is convergent of very short focal length. The ocular is of greater focal length, and is usually a Huyghenian eye-piece.

* The definition usually adopted for the *magnifying power of a microscope* is the ratio of the angle under which unit length of the object is seen to the angle under which it would be seen directly when placed at the least distance of distinct vision. It is therefore equal to $\frac{\Delta'}{f}\left(1+\frac{F_2E}{\Delta}\right)$. Also Δ' is taken conventionally as 10 inches, or as 25 cm. The microscope is then said to magnify so many diameters. But unless Δ be also 10 inches, which is not necessarily the case, this ratio is not really the linear magnification produced by the instrument; and although it is the estimate of the magnification naturally formed by an observer in the case of objects large enough to be distinguished by the naked eye, yet it depends both on Δ, the distance of vision he uses with the instrument, and on Δ', his actual minimum distance. For these reasons it seems preferable to use amplifying power, in comparing the intrinsic powers of two microscopes.

The whole may be regarded as one system of known focal length, but the objective is furnished with several oculars, to give the entire instrument various powers.

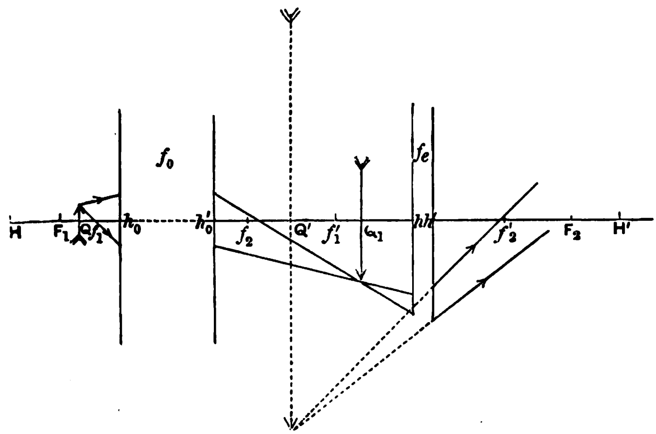


Fig. 69.

The objective, typified in the figure by f_o, will form an inverted and magnified image Q_1 of a small object placed beyond its first principal focus f_1. This image falls behind the first focus, f_1', of the ocular, typified by fe, and an image Q', further enlarged, is formed by that system.

If c be the distance between the second focus f_2 of the objective and the first focus f_1' of the ocular, the focal length of the entire instrument is $f_o f_e/c$. The entire system is divergent, having its foci between its unit points, and the object is placed just behind the first focus F_1.

Such a figure and the general explanation based on the principle of Images can only be regarded as correct for simple instruments with low magnifying powers.

In the best compound microscopes the rays from a point on the axis diverge at large angles with the axis, and pass through several lenses before incidence on the ocular, through which they pass, making comparatively small angles with the axis.

By the use of so many surfaces, whose radii are chosen appropriately, the rays from a point whether on the axis or

very near it are ultimately brought accurately to a focus, and that too for different values of the refractive index. In other words, the defects of Distortion and Chromatism are almost completely removed.

In an apochromatic objective of C. Zeiss, of Jena, of which the focal length is $\frac{1}{10}$th inch, are included the following lenses:—

(i) a simple frontal lens, greater than a hemisphere, which is made of flint and has a refractive index 1·72;

(ii) an achromatic lens formed of two simple lenses;

(iii) a simple lens of crown-glass;

(iv) an achromatic lens of three simple lenses;

(v) a correcting achromatic lens of three glasses.

In three of these lenses crown-glass is replaced by fluorite; also the medium between the front lens and the cover-glass of the object is a liquid of refractive index 1·65, while the slip and cover-glass are made of a medium with the same index as the front lens. The magnification obtainable varies with the ocular from 100 to 1800.

It is plain that the method of Images, and the approximations for Aberration and Distortion, which are carried only as far as the squares of the angles of divergence, are not really sufficient for such an instrument, where the rays may make an angle of nearly 80° with the axis. Practically the paths of rays inclined at finite angles to the axis must be worked out trigonometrically for different zones of the refracting surfaces, and different values of the refractive indices.

Lastly Abbe has shewn that with very high magnifying powers (2000—3000) the image seen of a veined or ridged structure is not even that suggested by Geometrical Optics. The latter is masked by diffraction images, which can only be explained and calculated on the Undulatory Theory, and the principles of Physical Optics*.

* Abbe, *Archiv f. Mikr. Anat.* vol. IX. p. 413 (1873). Rayleigh, *Phil. Mag.* vol. XLII. p. 167 (1896).

EXAMPLES.

1. The external focal length of a human eye is 15 millimetres, the internal focal length is 20 mm., and the first unit point is 2 mm. behind the front surface of the eye. A thin convergent lens is placed just in front of the eye so as to throw the final focus, for parallel rays falling on the lens, a distance 5 mm. before its position when the rays fall on the eye alone. Shew that the necessary focal length of the lens is 47 mm.

2. If the eye be considered as made of homogeneous refracting substance for which $\mu = 4/3$, and the distance from the retina to the front be invariable and equal to b, while the radius of curvature of the front surface can vary from a to $a(1-n)$, determine the focal length of the spectacles necessary to see distant objects when the radius of curvature is a; and shew that with these spectacles all objects at a distance $> 3a(1/n - 1)$ can be seen distinctly by means of the power of accommodation of the eye.

3. A ray is refracted into an eye and reflected from the anterior surface of the crystalline lens at a point distant y from the axis of the eye. Prove that if the curvature of the reflecting surface be increased by σ, the deviation of the emergent ray will be increased by $2y\sigma\{\mu - (\mu - 1)a/r\}$, where r is the radius of the cornea, a the distance from cornea to lens, and μ the index of the aqueous humour. Hence prove that the image of a candle produced by reflection at this surface will be diminished by increase in the curvature, provided $(\mu - 1)a < \mu r$.

4. A telescope consists of an objective of focal length F and a moveable eye-piece of focal length f. The distance between the principal foci of the objective is d_1, and between those of the eye-piece d_2. The eye is at distance d behind the second focus of the eye-piece, and the distance of distinct vision is Δ. Shew that the *angular* magnification of an object at distance u in front of the first focus of the objective is

$$-\frac{F}{f}\left\{1+\left(d_1+d_2+d-\frac{f^2}{\Delta-d}\right)\frac{1}{u}+\frac{F^2}{u^2}\right\}\left(1-\frac{d}{\Delta}\right).$$

5. A telescope consists of a single object-glass of focal length F and an eye-piece of focal length f. The eye is always placed at the eye-ring. Shew that if M_1 be the magnifying power when adjusted for a distance of distinct vision Δ_1, and M_2, Δ_2 be corresponding quantities,

$$(M_1\Delta_1 - M_2\Delta_2)F = (M_2{}^2\Delta_2 - M_1{}^2\Delta_1)f = M_1M_2(M_1 - M_2)\Delta_1\Delta_2.$$

6. Two collimators of the usual construction point directly towards each other, and the wires of each are made by adjustment to be seen distinctly with the images of the wires of the other, the numerical focal lengths being f and f'; δf and $\delta f'$ are small deviations of the positions of the wires from the geometrical foci, and D the interval between the object-glasses; shew that

$$\frac{\delta f}{f^2}+\frac{\delta f'}{f'^2}=\frac{\delta f\,.\,\delta f'}{f^2 f'^2}(D-f-f').$$

7. An astronomical telescope in normal adjustment has a stop of radius r placed at the focus to exclude all but full pencils. Shew that if p be the radius of the pupil, ρ the radius of the eye-ring, part of the field of view cannot be seen at all unless the distance of the eye from the eye-ring be less than $f(\rho+p)/r$, and that all the rays cannot enter the eye, unless its distance from the eye-ring be less than $f(p-\rho)/r$.

8. In an astronomical telescope, however adjusted, the angular radii of the field of view seen by full pencils and of the extreme field are Θ_1 and Θ_2. Shew that the extreme length visible of an object at distance u is $u\Theta_2-y_0$, and the length visible by full pencils is $u\Theta_1+y_0$.

Shew that, if a stop be placed at *the focus of the object-glass* of radius $F\Theta_1$ suitable for a star, the length visible by full pencils is $u\Theta_1-y_0$.

9. An astronomical telescope is fitted with a Huyghens' eye-piece and is in normal adjustment; shew that, if r be the radius of aperture of either lens of the eye-piece, R that of the object-glass, the angular radius of the field of view seen by full pencils is

$$3(2Fr-3fR)/F(2F+9f) \text{ or } (2Fr-3fR)/F(2F-3f),$$

as the magnifying power is less or greater than 3; f being the focal length of the eye-glass, F that of the object-glass.

10. An astronomical telescope is fitted with Ramsden's eye-piece, and is in normal adjustment. The ratio of the radius of either equal lens of the eye-piece to the focal length of that lens is m; that of the radius of the object-glass to its focal length is n; the magnifying power is M. Shew that the angular radius of the field of view seen by full pencils is the smaller of the two (positive) expressions $(4m-n)/(3M+1)$ and $(4m-3n)/(M+3)$.

The instrument is readjusted so that the final image of a distant object is at distance Δ from the eye, wherever that may be placed, and the magnifying power is now M'. Shew that the corresponding expressions for the field are

$$(4m-n+3nF/M'\Delta)/(3M+1-3F/M'\Delta)$$

and

$$(4m-3n+nF/M'\Delta)/(M+3-F/M'\Delta).$$

11. A stop of radius r is placed in the focus of the object-glass of an astronomical telescope in normal adjustment fitted with a Ramsden's eye-piece; shew that the radii of the effective apertures of the field-glass and eye-glass for rays from a star are respectively

$$\{R+(3M+1)r\}/3M \text{ and } \{3R+(M+3)r\}/3M,$$

where M is the magnifying power, and R is the radius of aperture of the object-glass.

12. A telescope in normal adjustment has an erecting eye-piece composed of two given eye-pieces of focal lengths f and f': shew that when the distance between the eye-pieces is so chosen that the length of the telescope from object-glass to eye-ring is least, the magnifying power is given by the equation

$$M^3-MF^2\left(\frac{1}{f^2}+\frac{1}{f'^2}\right)-2\frac{F^2}{ff'}=0.$$

13. An erector, consisting of two thin convergent lenses each of focal length f at a distance $2a$ apart, is inserted between the object-glass of a telescope and the first image in such a manner that the position of the latter is unaltered. Shew that the distances of the two lenses of the erector from the image are $a\{(a+f)/(a-f)\}^{\frac{1}{2}} \pm a$. Shew that the magnifying power is altered in the ratio

$$2a^2 - f^2 + 2a(a^2 - f^2)^{\frac{1}{2}} : f^2.$$

14. The focal length of the object-glass of an astronomical telescope is 40 inches, and the focal lengths of four convergent lenses forming an erecting eye-piece are respectively $\frac{3}{2}$, $\frac{1}{6}$, $\frac{3}{8}$, $\frac{3}{2}$ inches, reckoning from the field-glass. The interval between the first and second is one inch, between the second and third half an inch, and between the third and fourth is arbitrary. Shew that when the instrument is in normal adjustment, the magnifying power is 80/3, and the distance of the eye-lens from the object-glass is $41\frac{1}{2}$ inches.

15. A person views an object at distance u through an opera-glass so that the image is at a given distance from the eye, and then reverses the instrument.

Shew that the angular magnifications in the two cases are in the ratio

$$(F - l')(u - l') - f(u - F) \; : \; F(u + f) - (f + l)(u - l),$$

where l and l' are the lengths of the instrument in the two cases, F and f the focal lengths of the two lenses.

16. A Newtonian telescope is in normal adjustment, and the eye is placed at the eye-ring; shew that a person whose distance of distinct vision is Δ will need no readjustment of the telescope when looking at an object at distance $F^2\Delta/f^2$ from the object-mirror.

17. In a Newtonian telescope the ratio of the focal length of the mirror to its radius of aperture is 20 : 1, and the ratio of that of the eye-glass to its focal length 4 : 1. The distance between the centre of the plane mirror and the first focus of the eye-glass is equal to the radius of aperture of the concave mirror. Shew that the fraction of the incident light necessarily intercepted by the plane mirror is $\frac{1}{400}\left\{1 + \frac{76}{(M+1)}\right\}^2$, where M is the magnifying power.

18. In the Melbourne equatorial (Cassegrain's telescope) the focal lengths of the large and small mirrors are respectively 360 and 75 inches; the distance from the focus of the large mirror to the first focus of the eye-piece is also 360 inches. The focal length of the eye-piece is 8 inches.

Find the position of the eye-ring in normal adjustment; and shew that to place the instrument in adjustment for a distance of distinct vision 8 inches, the eye remaining at the same point as before, the small mirror must be moved inwards ·31 inch approximately, and the magnifying power decreases from 225 to 224·6.

19. Shew that in Cassegrain's telescope, when in normal adjustment with a single thin lens as eye-glass, the distance of the *optical centre* from the second focus of the eye-glass is

$$(a+2f_e+MFf_e/f_m)/(M^2-1).$$

With the numerical data of the previous question, shew that the distance between the eye-ring and the optical centre is ·00032... parts of an inch.

20. In Gregory's telescope the mirrors and eye-glass have focal lengths F, f_m and f_e, and apertures of radii y_o, y_m, and y_e; shew that the necessary size of y_m in order that no useful light may be lost is given by

$$\frac{y_m}{f_m}=\frac{y_e(F+f_e+x)+y_o(f_m+f_e+a+x)}{F(a+f_e+x)+xf_e},$$

where a is the distance between the focus of the large mirror and the first focus of the eye-glass, and $x(x+a)=f_m^2$.

Shew that the apparent angular radius of the field of view, excluding the ragged edge, is

$$\frac{y_e}{f_e}-\frac{y_m+y_e}{f_m+f_e+a+x}.$$

21. In Gregory's telescope the focal lengths of the mirrors are F and f_m, that of the eye-piece is f_e; and the distance in normal adjustment between the foci of the two mirrors is b. Shew that for an eye, placed at the second focus of the eye-piece, which sees distinctly at distance Δ, the small mirror must be moved inwards a distance equal to the lesser root of the equation

$$x^2-x(f_m^2/b+f_e^2/\Delta+b)+bf_e^2/\Delta=0;$$

and that the magnifying power is increased in the ratio $b : b-x$.

22. Shew that if a microscope formed of lenses whose relative positions are fixed be pushed nearer to the object by a small distance δ, the change in the position of the image is $m^2\delta$, where m is the linear magnification; and the change in the magnification is $m^2\delta/f$, where f is the focal length of the instrument.

23. A small object is viewed through a microscope and is placed at distance a from the front lens. The breadth of the pupil of the eye is p and y is the breadth at the front lens of a pencil which finally fills the pupil, the least distance of distinct vision is c; shew that the magnifying power is cy/pa.

CHAPTER VII.

DISPERSION AND ACHROMATISM.

119. WE have supposed hitherto that light is simple or homogeneous, and that in refraction from one medium to another, the relative index of refraction has a definite value independent of the nature of the light. But in reality the light of the sun and that of all incandescent bodies is heterogeneous, being composed of an infinite number of rays of homogeneous light differing in refrangibility. This was first proved by Newton.

The following was his first experiment with sunlight.

"In a very dark chamber, at a round hole, about one-third part of an inch broad, made in the shut of a window, I placed a glass prism, whereby the beam of the sun's light, which came in at that hole, might be refracted upwards toward the opposite wall of the chamber and there form a coloured image of the sun. The edge of the prism was perpendicular to the incident rays. The prism being placed in the position of minimum deviation I let the refracted light fall perpendicularly on a sheet of white paper, and observed the figure and dimensions of the solar image formed on the paper by that light. This image was oblong and not oval, but terminated with two rectilinear and parallel sides, and two semicircular ends. On its sides it was bounded pretty distinctly, but on its ends very confusedly and indistinctly, the light there decaying and vanishing by degrees. The breadth of this image answered to the sun's diameter and was about $2\frac{1}{8}$ inches including the penumbra. For the image was $18\frac{1}{2}$ feet from the prism, and at this distance that breadth, if diminished by the diameter of the hole, subtended an angle at the prism of about half a degree, which is the sun's apparent diameter. But the

length of the image was about $10\frac{1}{4}$ inches; and the refracting angle of the prism, whereby so great a length was made, was 64°. With a less angle the length of the image was less, the breadth remaining the same. This image or spectrum was coloured, being red at its least refracted end, and violet at its most refracted end, and yellow, green and blue in the intermediate spaces, which agrees with the proposition that lights which differ in colour, do also differ in refrangibility."

The effect of a second refraction was next investigated by Newton as follows:

"I placed a second prism immediately after the first in a cross position to it, so that it might again refract the beam of the sun's light which came to it through the first prism. In the first prism this beam was refracted upwards, and in the second sideways. And I found that by the refraction of the second prism, the breadth of the image was not increased, but the superior part, which in the first prism suffered the greater refraction and appeared violet and blue, did again in the second prism suffer a greater refraction than its inferior part, which appeared red and yellow, and this without any dilatation of the image in breadth."

From these and other experiments, which are fully described in his *Opticks*, Newton concluded that sunlight is composed of an infinite number of rays of homogeneous light, and that each of these possesses a definite refrangibility, and is not further decomposed by refraction.

The colours of the visible part of the solar spectrum were described by Newton as red, orange, yellow, green, blue, indigo, violet, beginning at the less refrangible end. The brightness is greatest in the yellow and adjacent colours, and decreases considerably towards the violet end. Besides the visible part the spectrum extends further on either side into the ultra-red and the ultra-violet; the heating effects of the sun's rays increase from the violet to the red, and are greatest in the ultra-red; the actinic effects are greatest in the ultra-violet.

120. The spectra obtained by these methods are made up of the overlapping coloured images of the sun's disc, and the colours are not separated. To obtain a pure spectrum we must diminish as much as possible these images.

This is most simply done by putting an achromatic lens between the slit A, at which the light is admitted, and the prism. If the lens form an image of the slit at B (Fig. 70) before the

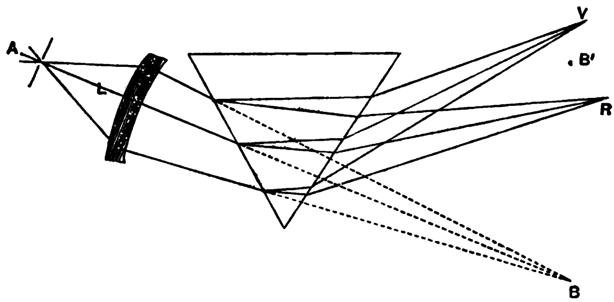


Fig. 70.

prism is introduced, and the prism be placed that the mean rays traverse it with minimum deviation, there will be formed different coloured images $RB'V$ of the slit at nearly the same distance from the edge as B. These images, arranged side by side, form the pure spectrum; and this spectrum may be received on a screen, or it may be viewed through an achromatic eye-piece.

For the purposes of spectrum analysis other apparatus would be used.

A heliostat reflects the sun's rays in a fixed direction so that they always fall on the slit. To obtain a sufficiently bright image a lens B is placed between the heliostat and the slit, so as to form an image of the sun on the slit. All the light that falls on the lens passes through the slit, for the sun's image will be practically a point, or if a cylindrical lens be used, a line of light parallel to the slit.

The light after passing the slit falls on a collimator C, *i.e.* an

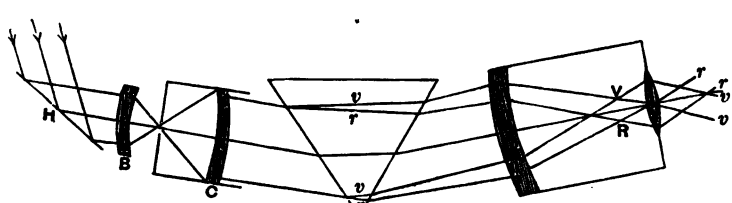


Fig. 71.

achromatic lens having its focus at the slit. By this means the

light will leave the collimator parallel to its axis and the angle of incidence on the prism will be the same for all the rays. Consequently on emerging from the prism the rays of the same colour will all be parallel; and the colours in the spectrum formed on a screen at a sufficient distance from the lens will be separated. The spectrum may also be viewed through a telescope; the rays of different colours will converge to separate points on the focal plane RV, and this spectrum may be viewed through an eye-piece.

When a pure solar spectrum is formed by these means it is found that instead of being a continuous band of colour, it is crossed by a number of dark lines. Light of certain definite refrangibilities is absent from the sun's rays, and the spaces in the spectrum at which these rays would have formed images of the slit are therefore black. Whatever the kind of glass of the refracting prism, the lines are always present in the same order, though the relative distances apart are variable. The principal lines in the visible solar spectrum are denoted by the letters A, B, C, D, E, F, G, H. The lines A, B, C are in the red, D is in the orange, E in the yellow end of the green, the most brilliant part of the spectrum, F in the blue end of the green, G in the indigo, and H in the violet.

There are a very great number of fixed lines. They were first observed by Frauenhofer in 1814, and are known by his name. Their positions have been determined, and the refractive indices corresponding to them for different materials obtained by measurements of the minimum deviation in a prism.

121. When light other than sunlight is used the spectrum changes its character. The spectra of incandescent solids and liquids are continuous, containing rays of all degrees of refrangibility from the extreme red to a higher limit depending on the temperature.

The spectrum of an incandescent gas or vapour on the other hand consists in general of a definite number of bright lines seen on a dark background. The constancy of the presence of these bright lines is the foundation of Spectrum Analysis. The existence of the dark lines in the solar spectrum is explained by the principle that a gas, emitting light of definite refrangibility, will absorb those rays. If then light containing all rays fall on a gas, the gas

absorbs from the light just those rays which it would itself emit if incandescent; consequently dark lines are seen in the spectrum of the light. If the gas be itself incandescent the absorption still takes place, but the light of the gas itself is substituted for the light absorbed. Thus if the light from the incandescent gas be more brilliant than the corresponding rays which have been absorbed, we see in the spectrum the bright lines of the gas itself. If the gas be less brilliant than the source of the white light, the rays emitted by the gas are less intense than those which it has absorbed and less intense than those in the immediate neighbourhood, and so relatively dark lines cross the spectrum.

For a fuller account of the principles of spectrum analysis, and also for effects known as anomalous dispersion, produced by passing light through certain substances, whereby the order of the colours is altered, the student is referred to Glazebrook's *Physical Optics*, and Roscoe's *Spectrum Analysis*.

122. Dispersive power.

The deviation produced by a prism or lens in any incident ray varies with the refrangibility of the ray. A ray of white light falling on a prism or lens will emerge as rays of different colours in different directions, and the angle between any two coloured rays is known as the *dispersion* for those two colours.

The dispersion produced for the same two colours by different substances varies with those substances; and the ratio of the dispersion to the mean deviation is called the *dispersive power* of the refracting medium used. As the formation of achromatic telescopes is of the greatest importance we may estimate the dispersive power of a refracting substance by considering the effect of a lens of that material on an incident ray.

Consider a thin lens of given power κ for some standard ray, taken preferably in the brightest part of the spectrum, and let the index of refraction for that ray be μ.

Then since $\kappa = (\mu - 1)(1/\rho - 1/\sigma)$, and since the deviation produced by the lens in a ray is κy, the dispersion of the two rays of indices μ and $\mu + \partial\mu$ is $y\partial\kappa$; and the ratio of this to the mean deviation is $\dfrac{\partial\kappa}{\kappa}$, which is equal to $\dfrac{\partial\mu}{\mu - 1}$.

This quantity, denoted by ϖ, is taken as the measure of the dispersive power of the substance for the two selected rays.

A pencil of white light diverging from a point on the axis and

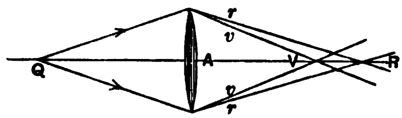


Fig. 72.

passing through a lens, will be broken into an infinite number of rays according to their refrangibilities, and the coloured foci of the pencil will be separate points on the axis extending from V to R.

The image formed on a screen placed at any point between R and V will therefore be not a white point, but a circle of confused colour. If the screen be placed at V the border of the image is reddish; if at R, the border will be violet.

If z be the distance from the lens of the focus for the mean ray, the rays of index $\mu + \partial\mu$ will pass through the boundary of a small circle of radius $z\,(y\partial\kappa)$, which is equal to $\varpi zy\kappa$ or $\varpi y\,(z/f)$.

This defect in the image formed by a lens is known as chromatic aberration, and owing to the magnitude of ϖ it is the most serious difficulty in the construction of refracting telescopes.

Newton formed the impression from an experiment made with a particular pair of substances that the dispersive power was the same for all substances. It would then be impossible to destroy the dispersion produced by two lenses close together, as in an object-glass, without destroying also the deviation; and therefore Newton constructed his reflecting telescope, while Huyghens designed refracting telescopes of very great focal length, 60 to 100 feet, in which the dispersion is less than in telescopes of shorter focal length. The fact that dispersive power varies with the substance was first discovered by a Mr Hall and rediscovered by Dollond, a London optician. The use of two lenses of crown and flint-glass enables us to construct an object-glass in which the dispersion is almost entirely destroyed without the deviation being destroyed; and on the other hand by using two prisms of crown-glass and one of flint-glass we can construct a

direct-vision spectroscope, in which the deviation for the mean ray is destroyed, but the dispersion of the different colours is not destroyed.

123. Irrationality of Dispersion.

When the dispersive powers of two materials for several pairs of selected rays are compared, it is found that the ratio of the dispersive powers varies with the rays selected.

This inequality of the ratios is known as *irrationality of dispersion.*

It follows that the spectra formed under similar circumstances by lenses or prisms of the two materials will not be geometrically similar at all points.

Hence, when the resulting dispersion is destroyed for a given pair of colours, by proper choice of the powers in the case of lenses, or of the angles in the case of prisms, there will be left outstanding a very small residuary dispersion for other colours. This gives rise to coloured images which are called secondary spectra.

Of recent years Professor Abbe, of Jena, has invented a whole series of glasses, made of other materials than silicates, in which the irrationality of dispersion has been overcome.

The following table* gives for four particular kinds of glass the refractive index for six lines of the spectrum, and the dispersive power.

The columns headed $\Delta\mu$ are the differences of consecutive μ's, reduced so that $\mu_H - \mu_B$ is represented by 1000 in each case, and illustrate irrationality of dispersion.

	HARD CROWN-GLASS.			EXTRA DENSE FLINT-GLASS.		
Line	μ	$\frac{\mu-\mu_e}{\mu_e-1}$	$\Delta\mu$	μ	$\frac{\mu-\mu_e}{\mu_e-1}$	$\Delta\mu$
B	1·5136	−·0129	50	1·6429	−·0225	43
C	1·5146	−·0111	132	1·6449	−·0194	121
D	1·5171	−·0062	168	1·6504	−·0110	159
E	1·5203		146	1·6576		144
F	1·5231	·0054	272	1·6642	·0100	280
G	1·5283	·0154	232	1·6770	·0294	253
H	1·5328	·0239		1·6886	·0470	

* Dr J. Hopkinson, *Proc. Roy. Soc.* 1877.

	Soft crown-glass.			Double extra dense flint.		
Line	μ	$\frac{\mu-\mu_e}{\mu_e-1}$	$\Delta\mu$	μ	$\frac{\mu-\mu_e}{\mu_e-1}$	$\Delta\mu$
B	1·5109	−·0137	49	1·7011	−·0251	43
C	1·5119	−·0118	131	1·7035	−·0217	119
D	1·5146	−·0066	167	1·7102	−·0124	157
E	1·5180		145	1·7191		143
F	1·5210	·0057	273	1·7272	·0121	281
G	1·5266	·0166	235	1·7432	·0335	257
H	1·5314	·0259		1·7578	·0537	

124. Achromatism of a system of lenses.

When a coaxial system of lenses in air is used to form the image of any object, the position and size of the image depend on the principal foci and the power of the system. Hence the various coloured images of any object can be made to coincide in magnitude and position for all colours and for all positions of the object, and perfect achromatism can be secured, only if the positions of the foci and the power be made independent of colour. We must therefore make $K, \frac{\partial K}{\partial \kappa_1}, \frac{\partial K}{\partial \kappa_n}$, with the notation of Art. 64, independent of colour; or by the identity $\frac{\partial K}{\partial \kappa_1}\frac{\partial K}{\partial \kappa_n} - K\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} \equiv 1$, any one of these conditions may be replaced by making $\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n}$ the same for all colours.

There must therefore be three quantities at our disposal, and consequently perfect achromatism cannot be secured with fewer than four lenses, if their focal lengths be given.

The equations for the distances between four lenses to which these conditions lead are given below, Art. 128. But their complexity forbids their use in practice, and the rule adopted for achromatism is that the power K be independent of colour, or if that be not possible, the same for two colours.

This may be justified as follows. The angle α' made with the axis of a system by an emergent ray, which before incidence made an angle α_0 with the axis and met the first surface at height y_1 is given by $\alpha' = y_1 K + \alpha_0 \frac{\partial K}{\partial \kappa}$.

Hence the angle $\partial\alpha'$ between the two emergent rays whose indices differ by $\partial\mu$ is $\left[y_1 \frac{\partial K}{\partial\mu} + \alpha_0 \frac{\partial}{\partial\mu}\left(\frac{\partial K}{\partial\kappa_1}\right)\right]\partial\mu$.

If this be made zero, the emergent coloured rays will be parallel and will produce in the eye practically the sensation of white light, the eye itself being very fairly achromatic. The relation thus obtained involves however y_1/α_0, and the system is achromatic, even to this limited extent, only for one position of the object. For other positions the first term in $\partial\alpha'$, which is by far the larger of the two, may be too large for the eye to disregard.

If then we make $\frac{\partial K}{\partial\mu}$ zero, the chromatic dispersion reduces to the second term, and, considering the necessarily small value of α_0, it may very well fall within the limits of clear vision for the eye.

With this rule only rays incident parallel to the axis emerge in parallel directions for all colours; the system is said to be achromatised for parallel rays.

125. Achromatic object-glasses.

Object-glasses are usually composed of two lenses, the first of which is a convergent lens of crown-glass, and the second a divergent lens of flint-glass. The dispersive power of flint-glass is nearly twice as much as that of crown-glass, so that the second lens can destroy the dispersion produced by the first, while the combination is convergent as a whole.

If these lenses be placed in contact, and if their thicknesses be disregarded, the deviation produced in any ray is Ky, where $K = \kappa_1 + \kappa_2$.

Hence, if K be made the same for two colours, the rays of those colours follow the same final path whatever the initial angle of divergence of the ray from the axis, and the images formed of those colours will coincide in magnitude and position for all positions of the object.

The condition that K may be the same for two colours, for which the dispersive powers of the materials of the lenses are ϖ_1 and ϖ_2, is that $\varpi_1\kappa_1 + \varpi_2\kappa_2 = 0$.

This equation shews that the lenses must be of opposite kinds; and it would be futile to take $\varpi_1 = \varpi_2$, as the power of the combina-

tion would then be zero. They must therefore be of different materials, and on account of irrationality of dispersion only two colours can be exactly combined. The resulting dispersions however for other colours will be extremely small compared with the dispersion produced by a single lens, and the chromatic aberration will be correspondingly reduced.

For visual purposes it is usual to combine two rays from the brightest parts of the spectrum, one from the reddish-yellow with one from the green-blue; the violet rays have then the largest outstanding dispersion, but they are much the faintest rays of the spectrum and the image appears practically white.

For photographic purposes two rays from the violet part of the spectrum, which is the actinic part, must be combined; and it is possible to unite the actinic rays more perfectly than the visible rays.

An object-glass is said to be over-corrected when the power of the flint-lens is too great in proportion to that of the crown-glass. The effect of this is to practically combine rays nearer the red end of the spectrum than is required and to increase the dispersion at the other end. For the ratio of the dispersive power of flint to that of crown increases on passing from the red to the violet end.

This error may be set right by slightly separating the lenses, which has plainly the same effect as a decrease in the power of the flint-lens.

For the total deviation is $\kappa_1 y_1 + \kappa_2 y_2$, and the crown-glass lens makes the rays converge so that y_2 is slightly less than y_1; this alteration may be regarded as a decrease in the factor κ_2.

But when the lenses are separated even to a slight degree the different coloured images no longer agree in position, and we ought in reality to take account also of the thicknesses, and make the total power the same for two colours (cf. Ex. 13, Chap. VII.).

The alteration produced in the relation between the powers must be very slight, as the ratios of the thicknesses to the focal lengths are generally extremely small. For example in the Yerkes objective at Chicago, the crown-glass lens is $2\frac{1}{2}$ in. thick at the centre, the flint-glass lens is $1\frac{1}{4}$ in. thick, and they are separated by an interval of $8\frac{3}{8}$ in. The focal length of the compound object-glass is 61 feet, and its radius of aperture 20 inches.

If we place n lenses in contact, the same argument as above shews that we can obtain the ratios of their powers from $(n-1)$ equations of the type $\Sigma\varpi\kappa = 0$, and so combine n colours.

Microscopic lenses are often composed of three lenses in contact; two plano-concave lenses of different kinds of flint-glass being cemented to a double-convex of crown-glass.

126. To find the dispersion in an achromatic object-glass.

Suppose the lines C and F united.

Then we have, taking C as the standard ray,

$$\varpi_C\kappa_1 + \varpi_C'\kappa_2 = 0, \text{ and } \frac{\kappa_1}{\varpi_C'} = \frac{\kappa_2}{-\varpi_C} = \frac{K}{\varpi_C' - \varpi_C}.$$

For any other ray λ, $K_\lambda = (\mu_\lambda - 1)(1/\rho - 1/\sigma) + (\mu_\lambda' - 1)(1/\rho' - 1/\sigma')$ and

$$K_\lambda - K_C = \varpi_\lambda\kappa_1 + \varpi_\lambda'\kappa_2.$$

Taking the values given for the first pair of glasses in the table, Art. 123, we find

$$(\kappa_1)_C = 2{\cdot}2318K_C, \quad (\kappa_2)_C = -1{\cdot}2318K_C$$

$$K_B - K_C = -{\cdot}0005K_C$$

$$K_D - K_C = +{\cdot}0003K_C$$

$$K_E - K_C = +{\cdot}0004K_C$$

$$K_G - K_C = -{\cdot}0019K_C$$

$$K_H - K_C = -{\cdot}0049K_C.$$

It is plain from the signs and values of these differences that the complementary colours are very closely united, and that the image is practically white.

The ratio $\kappa_1 : \kappa_2$ and the differences of the powers vary considerably with the nature of the glasses, and with the rays united.

127. Achromatic eye-pieces.

Since eye-pieces as a rule are composed of two convergent lenses, the achromatism that can be secured is only imperfect, and the rule that the power be independent of colour leads to the result, when the lenses are treated as thin, that the *distance between them is the arithmetic mean of their focal lengths.*

If the powers of the lenses be κ_1 and κ_2, and their distance apart be a, a ray that crosses the axis at a point distant u from the first lens and makes angle α_0 with the axis, will emerge making an angle α_2 with the axis given by the equation

$$\alpha_2/\alpha_0 = 1 + \kappa_1 u + \kappa_2(u + a) + \kappa_1\kappa_2 ua.$$

Hence if the lenses be of the same material, and ϖ be its dispersive power for a given pair of colours, the angle between the coloured emergent rays is $\partial\alpha_2$ where

$$\partial\alpha_2 = \varpi\alpha_0 \{u(\kappa_1 + \kappa_2 + 2\kappa_1\kappa_2 a) + a\kappa_2\},$$

the squares of ϖ being neglected.

If this be made zero, then the emergent coloured rays are parallel, or by Helmholtz's formula ($l'\alpha_2 = l\alpha_0$), the coloured images of a small object at distance u will all be of the same magnitude; their positions however will be separated by small intervals proportional to ϖ.

By taking the lenses of the same material the value of a is independent of ϖ, and therefore achromatism is secured to this extent for all colours. But the value of a obtained involves u and therefore the dispersion of rays from any other point on the axis may be large.

We therefore make the power independent of colour,

i.e. $$\kappa_1 + \kappa_2 + 2\kappa_1\kappa_2 a = 0 \text{ or } a = -\tfrac{1}{2}(f_1 + f_2),$$

and then the outstanding dispersion is $\varpi a\kappa_2\alpha_0$, and since in a telescope the rays in passing from the object-glass to the eye-piece make small angles with the axis, this residual dispersion may, for the centre of the field and for the brightest rays of the spectrum, be too small to be perceptible to the eye.

In Huyghens' eye-piece, where the focal lengths of the field-glass and eye-glass are $3f$ and f, their distance apart is $2f$, and the eye-piece is achromatic for parallel rays; in Ramsden's eye-piece, where the lenses are of equal focal length, the distance apart cannot be made equal to the focal length, as the first focus of the system would then fall at the field-glass, so that the latter would be visible with the image formed by the object-glass, and, moreover, no micrometer could be used.

With three lenses of the same material it is possible, by choosing the intervals apart so that K and $\dfrac{\partial K}{\partial\kappa_1}$ are independent of colour, to make the system achromatic, as far as the direction of the emergent rays, or the linear magnification of a small object, is concerned, for all positions of the origin of light.

128. Conditions of achromatism of four thin lenses of the same material.

The necessary and sufficient conditions for perfect achromatism are that

(i) the linear magnification of the object-glass A_1, which is $\left(\frac{\partial K}{\partial \kappa_1}\right)^{-1}$,

(ii) the linear magnification of the eye-glass A_4, which is $\frac{\partial K}{\partial \kappa_4}$,

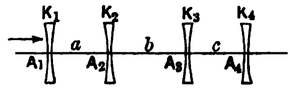


Fig. 73.

(iii) the apparent distance of A_4 from A_1, which is $\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_4}$,

must all be independent of colour (cf. Art. 124).

Hence by Cotes' formulæ,

$$1+\kappa_2 a+\kappa_3(a+b)+\kappa_4(a+b+c)+\kappa_2\kappa_3 ab+\kappa_2\kappa_4 a(b+c)+\kappa_3\kappa_4(a+b)c$$
$$+\kappa_2\kappa_3\kappa_4 abc \quad \text{.................(i)},$$

$$1+\kappa_3 c+\kappa_2(c+b)+\kappa_1(c+b+a)+\kappa_3\kappa_2 cb+\kappa_3\kappa_1 c(b+a)+\kappa_2\kappa_1(c+b)a$$
$$+\kappa_3\kappa_2\kappa_1 cba \quad \text{.................(ii)},$$

$$a+b+c+\kappa_2 a(b+c)+\kappa_3(a+b)c+\kappa_2\kappa_3 abc \quad \text{.................(iii)},$$

must all be independent of colour.

Differentiating these expressions with regard to μ, we obtain, since $\frac{\partial \kappa}{\partial \mu}=\varpi\kappa$, on neglecting squares and higher powers of ϖ, the three conditions

$$\kappa_2 a+\kappa_3(a+b)+\kappa_4(a+b+c)+2\kappa_2\kappa_3 ab+2\kappa_2\kappa_4 a(b+c)+2\kappa_3\kappa_4(a+b)c$$
$$+3\kappa_2\kappa_3\kappa_4 abc=0 \quad \text{.................(1)},$$

$$\kappa_3 c+\kappa_2(c+b)+\kappa_1(c+b+a)+2\kappa_3\kappa_2 cb+2\kappa_3\kappa_1 c(b+a)+2\kappa_2\kappa_1(c+b)a$$
$$+3\kappa_3\kappa_2\kappa_1 cba=0 \quad \text{.................(2)},$$

and
$$\kappa_2 a(b+c)+\kappa_3(a+b)c+2\kappa_2\kappa_3 abc=0 \quad \text{.................(3)}.$$

Multiply (3) by $\frac{1}{c}+2\kappa_4$ and subtract from (1); then

$$\kappa_4(a+b+c)-\kappa_2 ab/c=\kappa_2\kappa_3\kappa_4 abc \quad \text{.........................(4)}.$$

Similarly

$$\kappa_1(a+b+c)-\kappa_3 cb/a=\kappa_1\kappa_2\kappa_3 abc \quad \text{.........................(5)}.$$

Hence
$$\kappa_1\kappa_2 ab/c=\kappa_3\kappa_4 cb/a,$$

or
$$\kappa_1\kappa_2 a^2=\kappa_3\kappa_4 c^2 \quad \text{.............................(6)}.$$

If we eliminate b from (3) and (4), we obtain, using (6), the equation

$$\kappa_1+\kappa_2+\kappa_3+\kappa_4+2(\kappa_1\kappa_2 a+\kappa_3\kappa_4 c)+\kappa_1\kappa_2(\kappa_2+\kappa_3)a^2=0,$$

whence, by substitution from (6),

$$(\kappa_1+\kappa_2+\kappa_3+\kappa_4)/a=-\kappa_1\kappa_2-\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}\pm\sqrt{\kappa_1\kappa_2}R,$$

$$(\kappa_1+\kappa_2+\kappa_3+\kappa_4)/c=-\kappa_3\kappa_4-\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}\pm\sqrt{\kappa_3\kappa_4}R,$$

and on substituting in (3)

$$(\kappa_1+\kappa_2+\kappa_3+\kappa_4)/b=-\kappa_2\kappa_3\left(2+\frac{\kappa_1+\kappa_4}{\kappa_2+\kappa_3}\right)+\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}$$

$$\mp\sqrt{\kappa_2\kappa_3}\,R\,\frac{\sqrt{\kappa_1\kappa_3}+\sqrt{\kappa_2\kappa_4}}{\kappa_2+\kappa_3},$$

where $$R=2\sqrt{\kappa_1\kappa_2\kappa_3\kappa_4}-\kappa_1\kappa_3-\kappa_2\kappa_4-(\kappa_2+\kappa_3)^2.$$

129. Dispersion through a prism.

If a ray pass through a prism of refractive index μ, then with the usual notation in the case of a prism, we have $\sin\phi=\mu\sin\phi'$, $\sin\psi=\mu\sin\psi'$ and $\phi'+\psi'=i$.

Taking the total differentials of these equations,

$$\cos\phi\,\partial\phi=\mu\cos\phi'\,\partial\phi'+\sin\phi'\,\partial\mu$$

$$\cos\psi\,\partial\psi=\mu\cos\psi'\,\partial\psi'+\sin\psi'\,\partial\mu$$

$$\partial\phi'+\partial\psi'=0.$$

Eliminating $\partial\phi'$ and $\partial\psi'$ we have

$$\cos\phi\cos\psi'\,\partial\phi+\cos\phi'\cos\psi\,\partial\psi=\sin i\,.\,\partial\mu.$$

First, for monochromatic light, $\partial\mu$ is zero, and

$$-\frac{\partial\psi}{\partial\phi}=\frac{\cos\phi\cos\psi'}{\cos\phi'\cos\psi}.$$

This ratio of the small angle between two emergent rays to the angle between the incident rays may be called the magnifying power of the prism; we denote it by m.

Secondly, for two rays whose indices differ by $\partial\mu$, the dispersions before and after passing through the prism are connected by the equation

$$\partial\psi+m\,\partial\phi=\frac{\sin i}{\cos\phi'\cos\psi}\,\partial\mu.$$

For light incident in the same direction for all colours $\partial\phi=0$, and the dispersion is $\dfrac{\sin i}{\cos\phi'\cos\psi}\,\partial\mu$; where the angles ϕ', ψ are to be taken as those in the path of the mean ray.

130. Minimum dispersion in a prism.

We see that the dispersion $\partial\psi$ for a given pair of colours whose indices differ by a given quantity $\partial\mu$ is inversely proportional to $\cos\phi'\cos\psi$, and decreases as it increases.

Now if the angle of incidence of the ray, which is dispersed, be varied, its index remaining constant, we have on differentiating the equations which give the path of the ray in the prism

$$\cot\psi\, d\psi = \cot\psi'\, d\psi' = -\cot\psi'\, d\phi'.$$

Hence

$$\begin{aligned} d(\cos\phi'\cos\psi) &= -\sin\phi'\cos\psi\, d\phi' - \cos\phi'\sin\psi\, d\psi \\ &= (\sin\phi'\cos\psi - \cos\phi'\sin\psi\tan\psi\cot\psi')\, d\psi' \\ &= \cos\phi'\cot\psi'\cos\psi\,(\tan\phi'\tan\psi' - \tan^2\psi)\, d\psi'. \end{aligned}$$

The angles in the case of stationary dispersion are therefore connected by the equation

$$\tan^2\psi = \tan\phi'\tan\psi' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Since

$$\tan^2\psi = \mu^2\sin^2\psi'/(1-\mu^2\sin^2\psi') = \mu^2\tan^2\psi'/(\sec^2\psi' - \mu^2\tan^2\psi'),$$

this equation may be written

$$\mu^2\tan\psi' = \tan\phi'(\sec^2\psi' - \mu^2\tan^2\psi'),$$

i.e. $$\mu^2\tan\psi'(1+\tan\phi'\tan\psi') = \tan\phi'\sec^2\psi'$$

or $$\mu^2\sin\psi'\cos(\phi'-\psi') = \sin\phi'$$
$$= \sin(\phi'-\psi')\cos\psi' + \cos(\phi'-\psi')\sin\psi'$$

i.e. $$(\mu^2-1)\tan\psi' = \tan(\phi'-\psi') = \tan(i-2\psi') \ldots\ldots (2),$$

or $$(\mu^2-1)\tan(i-\phi') = \tan(2\phi'-i)\ldots\ldots\ldots\ldots\ldots(3).$$

These are cubic equations for $\tan\psi'$ and $\tan\phi'$, which will be shewn to have always three real roots; but a certain condition is necessary that one of the corresponding values of ψ' or ϕ' may lie between its extreme possible values α and $i-\alpha$, where α is the critical angle.

131. As the angle of refraction ϕ' decreases from its greates value α, when the incident ray grazes the first face of the prism, to its least value $i-\alpha$, when the emergent ray grazes the second face, the dispersion either decreases to a minimum and then increases, or it increases continually. This may be proved as follows.

We have, as before,

$$\begin{aligned}
d(\cos\phi'\cos\psi)\\
&= -\cos\psi\,\{\sin\phi' - \cos\phi'\cot\psi'\tan^2\psi\}\,d\phi'\\
&= -\cos\psi\left\{\sin(i-\psi') - \cos(i-\psi')\frac{\mu^2\tan\psi'}{\sec^2\psi' - \mu^2\tan^2\psi'}\right\}d\phi',\\
&= -\sec\psi\cos^3\psi'\,[(\sin i - \tau\cos i)\,\{1-(\mu^2-1)\,\tau^2\}\\
&\qquad\qquad - (\cos i + \tau\sin i)\,\mu^2\tau]\,d\phi',
\end{aligned}$$

where τ is written for $\tan\psi'$. The possible values of $\tan\psi'$ lie between $\tan(i-\alpha)$ and $\tan\alpha$.

First, let the angle of the prism, i, be acute. If we substitute in the cubic function of τ the values $-\infty$, 0, $\tan\alpha$ or $1/\sqrt{\mu^2-1}$ and $+\infty$ for τ, the signs obtained are $-, +, -, +$. Hence there are three real roots τ_1, τ_2, τ_3 in ascending order of magnitude. These values of τ are the roots of (2) Art. 130; and equation (3) has also real roots, since $\tan\phi'$ is connected linearly with $\tan\psi'$. Again, when ϕ' is α and ψ' is $i-\alpha$, the sign of the cubic function is that of $\sin\alpha - \cos\alpha\cot(i-\alpha)\tan^2\psi$. If $i<\alpha$ this is positive, and τ_1 is algebraically less than $\tan(i-\alpha)$; also τ_2 is between 0 and $\tan\alpha$.

But if $i>\alpha$ the cubic function is positive when $\phi'=\alpha$ and $\psi'=i-\alpha$, and therefore τ_2 will be greater than $\tan(i-\alpha)$,

if $\quad \sin\alpha\,\{\sec^2(i-\alpha) - \mu^2\tan^2(i-\alpha)\} > \mu^2\tan(i-\alpha)\cos\alpha,$

i.e. if $\quad \sin\alpha\,\{\sin^2\alpha - \sin^2(i-\alpha)\} > \sin(i-\alpha)\cos(i-\alpha)\cos\alpha$

or $\quad \sin\alpha\,\{\cos 2(i-\alpha) - \cos 2\alpha\} > \sin 2(i-\alpha)\cos\alpha$

i.e. if $$\sin(3\alpha-2i) - \sin\alpha\cos 2\alpha > 0 \quad\text{.............. (A).}$$

When this condition is satisfied (as it necessarily is when $i<\alpha$), τ_2 falls between $\tan(i-\alpha)$ and $\tan\alpha$; also we have

$$\begin{aligned}
&d(\cos\phi'\cos\psi)\\
&\quad= -\sec\psi\cos^3\psi'\cos i\,(\mu^2-1)\,(\tau-\tau_1)\,(\tau-\tau_2)\,(\tau-\tau_3)\,d\phi',
\end{aligned}$$

in which the factors $\tau-\tau_1$, $\tau-\tau_3$ are respectively positive and negative for possible values of τ, whether i be greater or less than α.

Hence the sign of the differential of the dispersion is that of $(-d\phi')\,(\tau-\tau_2)$, and as ϕ' decreases from α to $i-\alpha$, while ψ' increases from $i-\alpha$ to α, the dispersion decreases till $\tan\psi'=\tau_2$, and then increases.

If on the contrary the condition (A) be not fulfilled the root τ_2 is less than $\tan(i-\alpha)$ and therefore $\tan\psi'$ is always greater than τ_2, and the dispersion increases continually as the angle of incidence decreases and that of emergence increases.

Secondly, let the angle i be obtuse. Since no ray can pass unless $i < 2\alpha$, this requires that the critical angle $> \pi/4$ and that $\mu < \sqrt{2}$.

On giving τ the values $-\infty$, $-\sqrt{(\mu^2+1)/(\mu^2-1)}$, 0, $\tan\alpha$ in the cubic function we obtain the signs $+, -, +, -$.

Hence there are three real roots, τ_3 and τ_1 negative and τ_2 positive. Also in the expression for $d(\cos\phi'\cos\psi)$ the factors $\tau-\tau_3$, $\tau-\tau_1$ are now both positive but the factor $\cos i$ is negative.

Hence we obtain as before that if condition (A) be satisfied the root τ_2 is greater than $\tan(i-\alpha)$, and the dispersion decreases to a minimum and then increases, but if (A) be not satisfied the dispersion increases continually as the angle of incidence decreases and that of emergence increases.

When the ray passes with minimum deviation, the dispersion is necessarily increasing, and is therefore greater than the minimum dispersion, if that exists. For when $\phi' = \psi'$,

$$d(\cos\phi'\cos\psi) = (d\psi')\cos\phi'\cot\psi'\cos\psi(\tan^2\psi' - \tan^2\psi),$$

and the last factor of the right-hand side is negative, so that the dispersion is increasing with the angle of emergence.

Taking the value $\mu = 1{\cdot}5203$ for the line E and for hard crown-glass the value of the critical angle α is $41^\circ\,7'\,47''$ and condition (A) will be satisfied if $i < 59^\circ\,9'\,11''$; while with $\mu = 1{\cdot}6576$ for extra dense flint, $\alpha = 37^\circ\,6'\,19''$ and condition (A) is satisfied if

$$i < 50^\circ\,56'\,2''.$$

132. Dispersion through a battery of prisms.

If there be n prisms placed with their edges parallel and all turned the same way, the dispersion for two colours continually increases as the rays pass through the battery of prisms.

Let ϕ_1, ϕ_1', ψ_1, ψ_1', be the angles defining the passage of the mean ray through the first prism; ϕ_2, ϕ_2', ψ_2, ψ_2' through the second prism, and so on.

Let m_1 be the magnifying power and c_1 the dispersion factor for the first prism. Then

$$m_1 = \frac{\cos\phi_1 \cos\psi_1'}{\cos\phi_1' \cos\psi_1}, \quad c_1 = \frac{\sin i_1}{\cos\phi_1' \cos\psi_1}.$$

We have the equations (Art. 129)

$$\begin{aligned} m_1\, \partial\phi_1 + \partial\psi_1 &= c_1\, \partial\mu_1 \\ m_2\, \partial\phi_2 + \partial\psi_2 &= c_2\, \partial\mu_2 \\ &\cdots\cdots\cdots\cdots \\ m_n\, \partial\phi_n + \partial\psi_n &= c_n\, \partial\mu_n. \end{aligned}$$

Also $\psi_1 + \phi_2$ = inclination of the adjacent faces of the first and second prisms = constant.

Hence $\quad \partial\psi_1 + \partial\phi_2 = 0 = \partial\psi_2 + \partial\phi_3 = \ldots = \partial\psi_{n-1} + \partial\phi_n.$

Hence by addition we obtain

$$\partial\psi_n + m_1 m_2 \ldots m_n\, \partial\phi_1 = c_n\, \partial\mu_n + m_n\, c_{n-1}\, \partial\mu_{n-1} + m_n m_{n-1}\, c_{n-2}\, \partial\mu_{n-2} + \ldots + m_n\, m_{n-1} \ldots m_2\, c_1 \partial\mu_1.$$

As a rule the prisms are placed each in the position of minimum deviation for some definite ray, as this arrangement can be easily recovered, although when a collimator is used, it is not necessary to secure distinct vision. In this case all the m's are unity, and the total dispersion for an incident ray of white light is $\Sigma c\partial\mu$, *i.e.* the sum of the dispersions for the prisms separately.

133. Direct-vision spectroscope.

If a number of prisms of different materials be placed with their edges alternately in opposite directions, it is possible to choose their angles in such a way that the total deviation for a mean ray is zero, but the dispersion for other rays may be large. In this case, with the notation of the previous article $\psi_1 \sim \phi_2$ is constant; and we obtain for a ray incident undispersed, the equation

$$\partial\psi_n = c_n \partial\mu_n - m_n c_{n-1} \partial\mu_{n-1} + \ldots + (-)^{n-1} m_n\, m_{n-1} \ldots m_2 c_1 \partial\mu_1.$$

For example, consider a prism of flint-glass, placed with its faces in contact with or parallel to those of two equal prisms of

crown-glass. If the total deviation be zero, the mean ray crosses the second prism symmetrically. Hence

$$\phi_2' = \psi_2' = \tfrac{1}{2}i_2,$$

and
$$\sin\phi_1 = \mu_1 \sin\phi_1',$$

$$\mu_1 \sin\psi_1' = \mu_2 \sin\phi_2',$$

$$\phi_1' + \psi_1' = i_1,$$

and
$$\phi_1 = i_1 - \tfrac{1}{2}i_2,$$

since the paths of the ray in air and in the flint-glass are parallel.

From these equations it follows that

$$i_1 = \sin^{-1}\left\{\frac{\sin(i_1 - \frac{1}{2}i_2)}{\mu_1}\right\} + \sin^{-1}\left\{\frac{\mu_2 \sin\frac{1}{2}i_2}{\mu_1}\right\},$$

whence $\sin(i_1 - \frac{1}{2}i_2) = \sin i_1 \sqrt{\mu_1^2 - \mu_2^2 \sin^2\frac{1}{2}i_2} - \mu_2 \cos i_1 \sin\frac{1}{2}i_2$,

i.e. $(\mu_2 - 1)\cos i_1 \sin\frac{1}{2}i_2 = \sin i_1 \{\sqrt{\mu_1^2 - \mu_2^2 \sin^2\frac{1}{2}i_2} - \cos\frac{1}{2}i_2\}$.

Also in this case $\phi_3 = \psi_1$, $\psi_3 = \phi_1$, whence we find $m_3 = 1/m_1$ and $c_3 = c_1/m_1$, and as the ray traverses the middle prism with minimum deviation $m_2 = 1$.

The dispersion is therefore given approximately by the equation

$$\begin{aligned}\partial\psi_3 &= (2c_1\partial\mu_1 - c_2\partial\mu_2)/m_1\\ &= (2\sin i_1\,\partial\mu_1 - 2\sin\tfrac{1}{2}i_2 \cos\phi_1'\,\partial\mu_2)/\cos\phi_1\cos\psi_1'.\end{aligned}$$

134. Taking the values $\mu_1 = 1{\cdot}5203$, $\mu_2 = 1{\cdot}6576$ for the line E in crown and flint-glass respectively, I find that, if $i_2 = 90°$, the corresponding value of i_1 is 60° 40′. These values give $\phi_1 = 15°\ 40'$, $\psi_1' = 50°\ 26'\ 27''$, and a dispersion for the rays B and H, in the red and violet respectively, equal to 2° 48′, when calculated by the differential formula above.

The actual courses of the rays might be drawn from the following table:

	μ_1	μ_2	ψ_1'	ϕ_2'	ϕ_3'	ψ_3
B	1·5136	1·6429	50° 23′ 23″	45° 12′ 57″	49° 52′ 21″	16° 28′ 4″
E	1·5203	1·6576	50° 26′ 27″	45°	50° 26′ 27″	15° 40′
H	1·5328	1·6886	50° 31′ 10″	44° 28′ 44″	51° 49′	13° 38′ 24″

Also in each case $\phi_1' + \psi_1' = 60°\ 40' = \phi_3' + \psi_3'$, $\phi_2' + \psi_2' = 90°$.

The dispersion thus calculated is 2° 49′ 40″.

But much greater dispersion can be obtained by taking two pieces of crown and flint-glass whose indices are farther removed, and by choosing the angle of the flint prism so that at the refraction from flint into crown the angle of incidence approaches the critical angle*.

For instance, with the soft crown-glass and the double-extra-dense flint

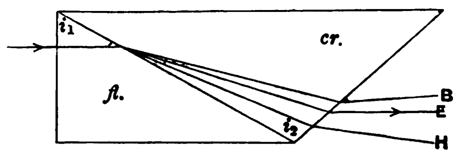


Fig. 74.

(Art. 123), a direct-vision spectroscope can be formed as in Fig. 74; the first prism is flint and the second crown-glass, and the light is incident normally on the first face.

If $i_1=60°$, $i_2=108° 8'$, the deviation is zero for the ray E,

$$\begin{array}{llll} \text{and for } B & \phi_2'=77° 10', & \psi_2'=30° 58', & \psi_2=51° 1'. \\ \text{,,} \quad E & \phi_2'=78° 45', & \psi_2'=29° 23', & \psi_2=48° 8', \\ \text{,,} \quad H & \phi_2'=83° 45', & \psi_2'=24° 23', & \psi_2=39° 13'. \end{array}$$

These two prisms give more than four times the dispersion of the two prisms of crown-glass enclosing one of flint.

EXAMPLES.

1. Shew that at a single refraction at a plane surface the dispersion is proportional to the tangent of the angle of refraction.

2. Shew that a pencil of light which is incident on one face of a triangular prism, and after entering the prism is reflected five times internally at the faces taken in order and emerges at the first face, will not be coloured.

3. An eye-piece is achromatised as far as regards the direction of the emergent rays for rays issuing from a given point on the axis. Shew that the radius of the circle of chromatic aberration of the emergent rays is $\varpi ya/f_1$, where y is the radius of aperture of the field-glass, f_1 its focal length and a the distance between the lenses.

4. A telescope with an achromatic object-glass and Huyghens' eye-piece is in normal adjustment. Shew that the greatest angle between the coloured emergent rays corresponding to an incident ray is $2\varpi\theta$, where θ is the angle subtended by the object-glass at its focus.

* Christie, *Proc. Roy. Soc.* 1877, vol. XXVI. p. 8.

5. An optical combination symmetrical about an axis is achromatic for all points in the region of the axis so far as the linear magnitudes of the images in two definite colours are concerned. Shew that the separation of the positions of the images in these two colours is constant, but that, if the achromatism in magnification only apply to very distant parts of the axis, the separation consists of a constant part and a part proportional to the square of the magnification.

In the case of Huyghens' eye-piece shew that the separation is $\varpi f(1+3m^2)$, where ϖ is the dispersive power, f the focal length of the eye-lens for the mean of the two colours, and m the magnification.

6. Shew that the coloured images of an object formed by an eye-piece of two lenses of the same material will be of the same apparent size if the distance apart of the lenses is equal to $-(f_1+f_2)/\{2+f_1/u+f_2/v\}$ where u and v are the distances of the object and the eye respectively from the instrument.

7. An eye-piece of two thin lenses of the same material brings the rays diverging from a point on the axis at distance u from the first lens to the same focus for all colours. Shew that the distance between the lenses must be $\dfrac{u}{u+f_1}(-f_1 \pm \sqrt{-f_1 f_2})$; and that the angle between the emergent coloured parts of the same incident ray is $\varpi y \dfrac{f_1 f_2 + u(f_2-f_1) \pm (2u+f_1)\sqrt{-f_1 f_2}}{f_1 f_2 (u+f_1)}$, where ϖ is the dispersive power of the material for the two colours, y is the radius of aperture of the first lens.

8. Two plano-convex lenses of thicknesses t and t' and of the same material are placed on the same axis. Shew that the combination is achromatic for rays parallel to the axis if their distance apart is $-\frac{1}{2}(f_1+f_2)$, their curved faces being towards each other.

Shew that if their plane faces be turned towards each other the condition of achromatism is

$$a = -\tfrac{1}{2}(f_1+f_2) - \frac{\mu+1}{2\mu^2}(t+t').$$

9. The condition for achromatism of a pencil of parallel rays passing directly through two spheres of radii r, r' with their centres at distance a apart is

$$\frac{\varpi}{\mu}\left(2a - \frac{\mu' r'}{\mu'-1}\right) + \frac{\varpi'}{\mu'}\left(2a - \frac{\mu r}{\mu-1}\right) = 0.$$

10. Three thin lenses of focal lengths f_1, f_2, f_3 and of the same material are placed on the same axis at distances a, b apart. Shew that the direction of the emergent light is the same for all colours and for all positions of the origin of light, if a and b are given by the equations

$$a^2(f_2+f_3) + 2af_2f_3 + f_2(f_2f_3+f_3f_1+f_1f_2) = 0$$
$$b^2(4f_1+f_2) + 2b(f_2f_3+2f_3f_1+2f_1f_2) + (f_2+f_3)(f_2f_3+f_3f_1+f_1f_2) = 0;$$

the ambiguities in the solutions of these equations for a and b being taken with opposite signs.

11. Prove that three thin lenses of numerical focal lengths f_1, f_2, f_3 in order, the first being divergent and the second and third convergent, for which $9f_1 = 2f_2 = f_3$, will give images of the same size for all colours, whatever the position of the object, provided that the distance from the first to the second be $3f_1$, and from the second to the third be $3f_3$.

The distance apart of the images of two colours, for which the dispersive power is ϖ, is constant and equal to $2\varpi f_3$.

12. Shew that if a coaxial system of thin lenses of the same material can give a perfectly achromatised image of a small object, the distance of the object from the first lens must be equal to each of the ratios $-\frac{\partial}{\partial\mu}\left(\frac{\partial K}{\partial\kappa_1}\right) \Big/ \frac{\partial K}{\partial\mu}$ and $-\frac{\partial}{\partial\mu}\left(\frac{\partial^2 K}{\partial\kappa_1\partial\kappa_n}\right) \Big/ \frac{\partial}{\partial\mu}\left(\frac{\partial K}{\partial\kappa_n}\right)$.

Three thin lenses of numerical focal length f, the first and second being convergent and the third divergent, are separated by intervals $2f$; shew that a perfectly achromatised image is formed of an object at distance $2f$ in front of the system.

13. Two thick lenses of materials of dispersive powers ϖ and ϖ' are placed at distance a apart. If t and t' be their thicknesses, K and K' their powers, and $\kappa_1, \kappa_2, \kappa_1', \kappa_2'$ the powers of their surfaces in order, the condition of achromatism for rays parallel to the axis is

$$\varpi\left[(1+\kappa_2' t'/\mu' + aK')(K+\kappa_1\kappa_2 t/\mu^2) + K'\kappa_1 t/\mu^2\right] \\ + \varpi'\left[(1+\kappa_1 t/\mu + aK)(K'+\kappa_1'\kappa_2' t'/\mu'^2) + K\kappa_2' t'/\mu'^2\right] = 0.$$

14. A double-concave lens of thickness μa is placed on the same axis between two thin lenses of powers κ_1 and κ_2, the distances between the thin lenses and the adjacent surfaces of the thick lens being both a. Shew that the system is completely achromatic for any object if the focal length of the thick lens is $\frac{\mu+1}{\mu-1}a$, and if

$$\left(\frac{1}{r_1}-\frac{1}{a}\right) \Big/ \mu a\kappa_2 = \left(\frac{1}{r_2}-\frac{1}{a}\right) \Big/ \mu a\kappa_1 = \frac{2(2\mu+1)}{(\mu^2-1)a} + \frac{1}{r_1} + \frac{1}{r_2},$$

where r_1 and r_2 are the radii of the surfaces of the thick lens, and all the lenses are of material of index μ.

15. A ray of white light is transmitted through a prism so that the extreme violet ray suffers minimum deviation. The light is received on a screen at distance h from the edge and perpendicular to the path of the violet ray in the prism. Shew that the breadth of the spectrum on the screen is

$$\frac{\mu_v-\mu_r}{\mu_v}\sec^2\frac{D}{2}\left[(2h-a)\tan\frac{D+i}{2} + a\cos\frac{D}{2}\cos\frac{D+i}{2}\tan\frac{i}{2}\sec\frac{i}{2}\right],$$

when D is the minimum deviation, and a the length of the path in the prism of the violet ray.

16. If a beam of parallel rays pass through a train of prisms of the same material, the chromatic dispersion for two colours, whose indices differ by $\partial\mu$, is proportional to the ratio of the difference in the thickness of glass traversed by the two sides of the beam to the breadth of the emergent beam.

17. Shew that if the index of refraction of the material of a prism be $\sqrt{2}$, the dispersion for a ray is a minimum when $\phi' = \frac{2}{3}i$, provided that $i < 67\frac{1}{2}°$.

18. A ray is incident normally on the first face of a direct-vision spectroscope composed of two prisms with their faces in contact. Prove that the relation between their angles i_1 and i_2 for zero deviation is

$$(\mu_1 - 1)\cot i_2 = \sqrt{\mu_2^2 \operatorname{cosec}^2 i_1 - \mu_1^2} - \cot i_1,$$

and that the dispersion is

$$(\sin i_2\, \partial\mu_2 \sim \cos\psi_2' \sin i_1\, \partial\mu_1)/\cos\psi_2 \cos\phi_2',$$

where ϕ_2', ψ_2', ψ_2 define the path of the mean ray in the second prism.

19. In a spectroscope consisting of a single prism with collimator and telescope, the latter two being fixed, and different parts of the spectrum being brought into the field of view by turning the prism, prove that for focusing two colours of refractive indices μ and $\mu + \partial\mu$, the prism must be turned through an angle $\Delta/(m-1)$, where Δ is the dispersion of the colours and m the magnifying power of the prism.

CHAPTER VIII.

ABERRATION.

135. IT was shewn in Chap. III., that any pencil of rays which diverges from a point on the axis of a system of spherical refracting surfaces, will finally form a pencil passing through a definite point on the axis, *provided that the squares of the angles of divergence can be neglected throughout, and that the incidence of the axis of the pencil be always direct.*

But if the angles of divergence, though small, be such that their squares must be retained, all the rays of such a pencil, for which the initial or final angles of divergence do not exceed certain definite limits, will finally touch a small part of a surface of revolution, called a *caustic* surface.

If the extreme rays finally cut the axis in Q', and if Q'_0 be the geometrical focus, the distance Q'_0Q' is called the *longitudinal aberration,* while the radius of the pencil as it passes Q'_0 is called the *lateral aberration.*

Let the final inclination of a ray to the axis be α, then it is clear that this ray cuts the axis at a distance from the geometrical focus proportional, as a first approximation, to α^2.

Let this distance be $a\alpha^2$, where a is a function of the position of the origin of light on the axis, and of the powers and distances apart of the refracting surfaces, which will be determined below in certain cases.

Hence, taking the geometrical focus as origin and the axis as axis of x, the equation of an emergent ray is $y = (x - a\alpha^2) \tan \alpha$, and differentiating to find the envelope, we obtain $x = 3a\alpha^2$, $y = 2a\alpha^3$ for the coordinates of the point of contact to this order of approximation.

The caustic surface is therefore generated by the revolution about the axis of x of the semi-cubical parabola $27ay^2 = 4x^3$. This surface will only extend from its cusp at the geometrical focus to the circular section given by $x = 3a\alpha'^2$, where α' is the inclination to the axis of the extreme ray; so that the greatest abscissa is three times the longitudinal aberration.

The aberration is usually reckoned positive if Q'_0Q' fall in the same direction as the emergent light; the cusp of the caustic then points in the opposite direction to that of the emergent light, and the refracting system is said to be *over-corrected*. When the cusp of the caustic points in the same direction as the emergent light, the aberration is negative, and the refracting system is said to be *under-corrected*.

136. Least Circle of Aberration.

The emergent pencil touches part of a caustic surface, with a

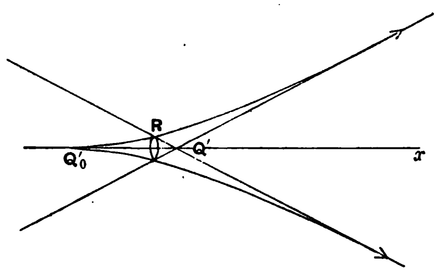


Fig. 75.

cusp at Q'_0, and will meet a screen placed perpendicularly to the axis in circles of different radii.

If the screen be placed at Q'_0 the appearance is that of a very bright point, surrounded by a halo of rapidly diminishing brightness. It is not possible to determine by the methods of geometrical optics the exact law of this illumination, when the screen is at Q'_0, or in any other position. For the rays that may unite at any point of the screen will have taken different optical paths and may be in phases which interfere with each other. Hence this is necessarily a question of physical optics.

But as the screen is moved from Q'_0 towards Q', the smallest area in which the light meets the screen occurs at some point R,

and is bounded by the points in which the extreme rays cut the caustic.

This circle is called the *Least Circle of Aberration*, and its smallness may be regarded as a test of the working of an optical instrument, *at least for very small objects on the axis.*

To find the coordinates of R, substitute those of any point on the caustic, $x = 3a\alpha^2$, $y = 2a\alpha^3$, in the equation of the extreme ray $y = \alpha'(x - a\alpha'^2)$, where we have written α' for $\tan\alpha'$, since the coordinate x is of the second order of small quantities.

This gives the cubic $2\alpha^3 = \alpha'(3\alpha^2 - \alpha'^2)$, which has two coincident roots $\alpha = \alpha'$, and the third root, which we require, is $-\frac{1}{2}\alpha'$.

Hence the abscissa of the centre of the circle of least aberration is $\frac{3}{4}a\alpha'^2$; *i.e.*, $\frac{3}{4}$ths of the longitudinal aberration; and the radius of the circle is $\frac{1}{4}a\alpha'^3$ *i.e.* $\frac{1}{4}\alpha'$ (longitudinal aberration).

137. The value of a' to be adopted for the extreme rays of the emergent pencil may be determined by the following considerations.

The origin of light being at a given point on the axis of the refracting system, the initial angle of divergence a_0, of the rays which ultimately, if not intercepted, pass through the boundary of any of the lenses or diaphragms, or of the pupil of the eye, is given by the equation

$a_0 =$ (Radius of aperture)/(Apparent distance of lens from the origin of light).

The least value of a_0 thus determined marks the boundary of the pencil every ray of which traverses the system and enters the eye. The corresponding value of a' may then be found by Cotes' formula; or we may equally well calculate the values given for a' by the equation

$a' =$ (Radius of aperture)/(Apparent distance of lens from the image),

and take the least value so found.

When the origin of light is at infinity on the axis, and the incident rays form a pencil parallel to the axis, the emergent rays, to the first approximation, pass through the second focus F_2, and the value of a' to be chosen is the least value obtained for the ratio

(Radius of aperture)/(Apparent distance of lens from F_2).

In the case of parallel rays we shall find that when the object glass governs the pencil, no part of which is cut off by the boundaries of the other lenses or of the pupil of the eye, the value of a' is

(Radius of object-glass)/(Focal length of system);

when the radius of the pupil governs the pencil, the value of a' is

(Radius of the image of the pupil)/(Focal length of system).

138. Aberration at reflection at a spherical mirror.

Let O be the centre and A the vertex of a spherical mirror,

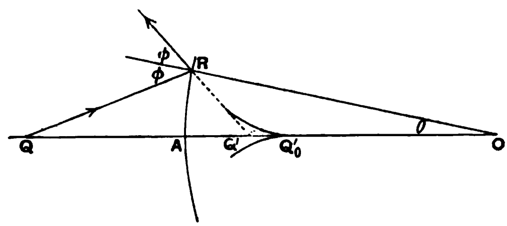


Fig. 76.

and let a ray from a point Q on the axis, after reflection at R, cut the axis in Q'.

We have, denoting OQ by p, OQ' by q and OA by r,

$$r/p = \sin(\phi - \theta)/\sin\phi\,; \quad r/q = \sin(\phi + \theta)/\sin\phi.$$

Hence $$1/p + 1/q = (2\cos\theta)/r \quad\text{..................... (i).}$$

But for the geometrical focus, we obtain, on putting θ zero,

$$1/p + 1/q_0 = 2/r \quad\text{.......................(ii).}$$

Therefore $$1/q_0 - 1/q = 2(1 - \cos\theta)/r \text{......... accurately (iii).}$$

If, however, only squares of the aperture be retained, we see that the longitudinal aberration $q - q_0$ is equal to

$$q_0^2\theta^2/r^2 \text{ or } q_0^2(q_0 - r)^2\alpha'^2/r^3.$$

The direction of the aberration Q'_0Q' is therefore the same as that of OA, and the cusp of the caustic points, whether the mirror be convex or concave, in the same direction as the curvature of the mirror.

139. Aberration in Cassegrain's telescope.

When a pencil of parallel rays is reflected at the object-mirror of Cassegrain's telescope, the aberration is directed from the centre towards the vertex, and the reflected rays cut the axis (virtually) in points to the right of the focus F (cf. Fig. 66, Art. 112). The aberration after reflection at the small convex mirror is in the same direction, and the rays cut the axis to the right of f_1, the focus of the eye-piece. But the displacement of the points where the reflected rays first cut the axis being from the focus f of the convex mirror, their conjugate points are nearer to f than is f_1, which is conjugate to F in the convex mirror. Hence the two aberrations tend to correct each other.

In Gregory's telescope, however, where the small mirror is concave, the second aberration is in the opposite direction, and therefore the displacement of F to the right and the second aberration unite to bring the focus to the left of f_1.

We see then that *for rays parallel to the axis* the aberration produced is less in Cassegrain's telescope than in Gregory's; the discussion of the effects of the curvatures of the mirrors on rays incident at a small angle with the axis would require us to determine both distortion and indistinctness.

The general expressions for these are given below in Chapter XIV.; and the results involve both the radius of aperture of the object-mirror and the angular field of view; but as most large telescopes give a comparatively small field, the second is the less important. The image formed by the telescope in the plane of the micrometer at f_1 will be therefore fairly distinct all over if it be so at the centre, *i.e.* if the aberration be small there.

When the distortion is calculated, the two chief terms in the expression are found to be of opposite signs in Cassegrain's telescope, and of the same sign in Gregory's. The field is also flatter in the former.

140. Aberration at refraction at a plane surface.

Let Q be an origin of light on the axis QA, and let a ray making an angle α with the axis be refracted at a plane surface from a medium of index μ into one of index μ'.

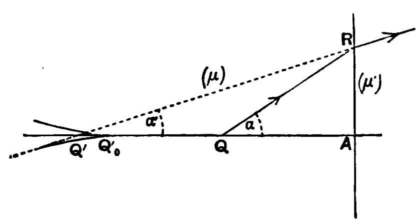


Fig. 77.

If the refracted ray cut the axis in Q', and α' be its angle of divergence, we have

$$\mu \sin \alpha = \mu' \sin \alpha'.$$

Also $$Q'A/QA = \cot \alpha'/\cot \alpha = \mu' \cos \alpha'/\mu \cos \alpha.$$

For the geometrical focus Q'_0,

$$Q'_0A/QA = \mu'/\mu.$$

Hence the longitudinal aberration Q'_0Q' is accurately

$$\frac{\mu' (\cos \alpha - \cos \alpha')}{\mu \cos \alpha} QA;$$

or is, if only squares of the aperture be retained,

$$\left\{\frac{1}{2}\frac{\mu'(\mu^2-\mu'^2)}{\mu^3}QA\right\}\alpha'^2.$$

The aberration is positive if $\alpha' > \alpha$, *i.e.* if the rays be refracted from a denser into a rarer medium; and negative if refracted from the rarer into the denser medium, as in the figure.

It is shewn later that the complete caustic surface, enveloped by the refracted rays, is generated by the revolution of the evolute of a conic about the axis; the conic being an ellipse in the first case, and a hyperbola in the second.

141. Aberration at a single refraction at a spherical interface.

Let a ray from Q be refracted at the point R of the spherical

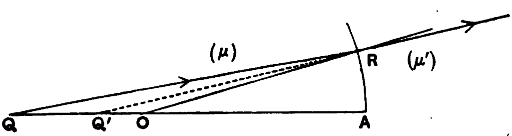


Fig. 78.

interface from a medium of index μ into one of index μ'; then if we denote AQ by u, AQ' by v and AO by ρ, and if θ be the angular aperture AOR, we have

$$\mu \sin QRO = \mu' \sin Q'RO,$$

or $$\mu OQ/RQ = \mu' OQ'/RQ' \text{ ....(i).}$$

Hence

$$\mu(u-\rho)\{\rho^2+(v-\rho)^2+2\rho(v-\rho)\cos\theta\}^{\frac{1}{2}}$$
$$=\mu'(v-\rho)\{\rho^2+(u-\rho)^2+2\rho(u-\rho)\cos\theta\}^{\frac{1}{2}}$$

or

$$\mu(u-\rho)v\left\{1-\frac{4\rho(v-\rho)}{v^2}\sin^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}}$$
$$=\mu'(v-\rho)u\left\{1-\frac{4\rho(u-\rho)}{u^2}\sin^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}}.$$

This equation is satisfied for all values of θ if u and v are given by the equations

$$\mu(u-\rho)v=\mu'(v-\rho)u,\quad \mu/v=\mu'/u;$$

i.e. if $$\mu'v=\mu u=(\mu+\mu')\rho.$$

The points so determined are the aplanatic points I and I' of Art. 41.

But in the general case, we have, on retaining only the squares of θ, and expanding the square roots, the equation

$$\mu(u-\rho)v\left\{1-\tfrac{1}{2}\frac{\rho(v-\rho)}{v^2}\theta^2\right\}=\mu'(v-\rho)u\left\{1-\tfrac{1}{2}\frac{\rho(u-\rho)}{u^2}\theta^2\right\};$$

or

$$\mu' u\rho-\mu v\rho=(\mu'-\mu)uv+\tfrac{1}{2}(\mu/v-\mu'/u)(u-\rho)(v-\rho)\rho\theta^2 \ldots\text{(ii)}.$$

On dividing by $uv\rho$ and putting $\rho\theta=y$, the radius of aperture, we obtain, as the second approximation, the equation

$$\mu'/v-\mu/u=(\mu'-\mu)/\rho+\tfrac{1}{2}(\mu/v-\mu'/u)(1/\rho-1/u)(1/\rho-1/v)y^2 \text{ (iii)}.$$

But the position of the geometrical focus of an origin of light at distance u_0 from A is given by the first approximation,

$$\mu'/v_0-\mu/u_0=(\mu'-\mu)/\rho \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Hence if the pencil is already affected with aberration due to previous reflections or refractions at surfaces on the same axis, so that Q is the origin of only a cone of rays and not of a full pencil, and if we put

$$v=v_0-\mathbf{d}v,\quad u=u_0-\mathbf{d}u,$$

we obtain on subtraction

$$\mu'\mathbf{d}v/vv_0-\mu\mathbf{d}u/uu_0=\tfrac{1}{2}(\mu/v-\mu'/u)(1/\rho-1/u)(1/\rho-1/v)y^2 \ldots\text{(v)}.$$

Multiply throughout by y^2, then since $y=u\alpha=v\alpha'=\rho\theta$, where α and α' are the angles of divergence, we have, to this order of approximation,

$$\mu'\alpha'^2\mathbf{d}v-\mu\alpha^2\mathbf{d}u=\tfrac{1}{2}(\mu\alpha'-\mu'\alpha)(\theta-\alpha)(\theta-\alpha')y \ldots\text{ (vi)}.$$

But in the terms on the right-hand side we may substitute the first approximation,

$$(\theta-\alpha)/\mu'=(\theta-\alpha')/\mu=(\alpha'-\alpha)/(\mu'-\mu),$$

whence we obtain the relation

$$\mu'\alpha'^2\mathbf{d}v-\mu\alpha^2\mathbf{d}u=\tfrac{1}{2}\left(\frac{\mu\mu'}{\mu'-\mu}\right)^2\left(\frac{\alpha'}{\mu'}-\frac{\alpha}{\mu}\right)(\alpha'-\alpha)^2 y \ldots\text{ (A)}.$$

In this formula the aberrations $\mathbf{d}u$, $\mathbf{d}v$ are reckoned positive if measured from the geometrical foci in the direction in which the light is travelling, and the incident pencil is supposed to be already

affected by aberration, indicated by the term $\mathbf{d}u$. This is the fundamental formula in calculating the aberration due to any coaxial refracting surfaces.

142. Example.

A pencil of light diverging from a point Q on the axis is refracted at a spherical interface. Shew that the radius of the least circle of aberration of the refracted pencil is $\frac{1}{8}\left(\frac{\mu}{\mu'}\right)^2 \frac{y^3}{\rho^2} \frac{IQ \, . \, OQ^2}{F_1Q \, . \, AQ^2}$.

The light diverging from a point Q, we have, by (A),

$$\mu' a'^2 \mathbf{d}v = \frac{1}{2}\left(\frac{\mu\mu'}{\mu'-\mu}\right)^2 \left(\frac{a'}{\mu'} - \frac{a}{\mu}\right)(a'-a)^2 y;$$

also the radius of the least circle of aberration is $\frac{1}{4}a' \, . \, \mathbf{d}v$.

If we substitute from the formula $\mu' a'/\mu a = 1 + \kappa u/\mu = \frac{\kappa}{\mu} F_1 Q$ (Art. 48), we obtain

$$\text{(i)} \quad a' - a = a\left\{\frac{\mu}{\mu'}\left(1 + \frac{\kappa u}{\mu}\right) - 1\right\} = a\frac{\kappa}{\mu'} OQ,$$

since $a' = a$ only when Q is at O.

$$\text{(ii)} \quad \frac{a'}{\mu'} - \frac{a}{\mu} = a\left\{\frac{\mu}{\mu'^2}\left(1 + \frac{\kappa u}{\mu}\right) - \frac{1}{\mu}\right\} = a\frac{\kappa}{\mu'^2} IQ,$$

since the aberration vanishes in this manner only when Q is at I.

Hence the radius of the least circle of aberration

$$= \frac{1}{8}\left(\frac{\mu\mu'}{\mu'-\mu}\right)^2 \left(a\frac{\kappa}{\mu'^2} IQ\right)\left(a\frac{\kappa}{\mu'} OQ\right)^2 \frac{y}{a\kappa F_1 Q} = \frac{1}{8}\left(\frac{\mu}{\mu'}\right)^2 \frac{y^3}{\rho^2} \frac{IQ \, . \, OQ^2}{F_1 Q \, . \, AQ^2},$$

since $\kappa = (\mu' - \mu)/\rho$ and $a = y/AQ$.

143. Aberration of a pencil refracted at coaxial spherical surfaces.

Let there be n spherical surfaces separating media of indices $\mu_0, \mu_1 \ldots \mu_n$, and let the origin of light be a point on the axis in the first medium. By the addition of n difference-equations of the type (A), we obtain the formula

$$\mu_n \alpha_n^2 \mathbf{d}v = \frac{1}{2} \sum_{r=1}^{r=n} \left(\frac{\mu_r \mu_{r-1}}{\mu_r - \mu_{r-1}}\right)^2 \left(\frac{\alpha_r}{\mu_r} - \frac{\alpha_{r-1}}{\mu_{r-1}}\right)(\alpha_r - \alpha_{r-1})^2 y_r \ \ldots\text{(i)}.$$

In this equation $\mu_r \alpha_r/\mu_0 \alpha_0$ or $\mu_r \alpha_r/\mu_n \alpha_n$ can be written down immediately by Cotes' formulæ (Art. 75), as linear functions of the distance u of the origin of light from the first surface, or of the distance v of the geometrical focus from the last surface respectively.

Also y_r/α_0 or y_r/α_n, the apparent distances of the rth surface from the origin or the focus, are linear functions of the same quantities u and v (Art. 89).

Hence the aberration is as a rule equal to the square of the initial angle of divergence α_0, multiplied by a quartic function of u and by the square of the linear magnification; or to the square of the final angle of divergence α_n multiplied by a quartic function of v. Consequently a system of coaxial spherical surfaces can be made aplanatic only for at most four isolated points on the axis, and not for any finite portion of it.

The case of rays entering parallel to the axis is easily included by putting $y_1 = \alpha_0 u$, making u infinite and α_0 zero. By substituting the values of $\mu_r\alpha_r$ and y_r we can obtain in this case the formula for the aberration at the second focus of the system

$$\mathbf{d}v = \tfrac{1}{2}\frac{\mu_n y_1^2}{K_n^2}\sum_{r=1}^{r=n}\left(\frac{\mu_r\mu_{r-1}}{\mu_r-\mu_{r-1}}\right)^2\left(\frac{K_r}{\mu_r^2}-\frac{K_{r-1}}{\mu^2_{r-1}}\right)\left(\frac{K_r}{\mu_r}-\frac{K_{r-1}}{\mu_{r-1}}\right)^2\frac{\partial K_r}{\partial\kappa_r}\ \ldots\text{(ii)},$$

where K_r is the power of the first r surfaces, $(K_0 \equiv 0)$.

One other important use of this formula may be mentioned. If the origin of light be at the first principal focus of the entire system, then to the first approximation the emergent rays are all parallel to the axis. The last of the difference-equations (A) must then be replaced by the equation from which it was derived; for $\mu'\mathbf{d}v/vv_0$ or $\mu'(1/v - 1/v_0)$ in equation (v), Art. 141, is here simply μ_n/v_n (writing μ_n for μ'), and therefore on multiplication by y_n^2, we obtain $\mu_n y_n^2/v_n$, which we may write as $\mu_n y_n(\mathbf{d}\alpha_n)$ where $\mathbf{d}\alpha_n$ will be the angle that the extreme emergent rays make with the axis. By substitution for $\mu_r\alpha_r$ in terms of α_n we can obtain the formula

$$\mu_n\mathbf{d}\alpha_n = \tfrac{1}{2}y_n^3\sum_{r=1}^{r=n}\left(\frac{\mu_r\mu_{r-1}}{\mu_r-\mu_{r-1}}\right)^2\left(\frac{K'_r}{\mu_r^2}-\frac{K'_{r-1}}{\mu^2_{r-1}}\right)\left(\frac{K'_r}{\mu_r}-\frac{K'_{r-1}}{\mu_{r-1}}\right)^2\frac{\partial K'_r}{\partial\kappa_r}\ldots\text{(iii)}$$

where K'_r is the power of the surfaces from the rth to the nth inclusive.

In this formula y_n is the radius of the pencil at the last surface, *i.e.* either the radius of the eye-ring or the radius of the pupil, the values to be given to the extreme apertures being governed by the methods of Art. 137.

144. We may also express the aberration at a spherical interface in terms of the distances of the conjugate foci from the centre.

Let $OQ = p$, $OQ' = q$, $OA = r$ (Fig. 78); where p, q and r are algebraic quantities, reckoned positive, if measured from the centre in the direction *opposite* to that of the incident light.

(With this convention it is clear that the distance *from Q to O* will be a positive quantity, when measured from the origin of light with the light, which is the direction of positive measurement that has been used throughout, when an origin of light is the origin of measurement.)

The aberration may be written as $\mathbf{d}v$ or $\mathbf{d}q$, and is given by the formula (A)

$$\mu'\alpha'^2\mathbf{d}q - \mu\alpha^2\mathbf{d}p = \tfrac{1}{2}\left(\frac{\mu\mu'}{\mu'-\mu}\right)^2\left(\frac{\alpha'}{\mu'} - \frac{\alpha}{\mu}\right)(\alpha'-\alpha)^2 y.$$

Let the radii of the incident and refracted pencils as they pass the centre O be denoted by η and η'; then $\eta = p\alpha$, $\eta' = q\alpha'$; and by Helmholtz's theorem, $\mu\eta = \mu'\eta'$, since the centre is the nodal point.

Hence $$\alpha' - \alpha = \mu\eta\left(\frac{1}{\mu' q} - \frac{1}{\mu p}\right) = \mu\eta\left(\frac{1}{\mu'} - \frac{1}{\mu}\right)\frac{1}{r},$$

by the relation for the geometrical focus (II. Art. 38).

Also in Fig. 78 OA is a negative quantity, and therefore

$$y = -r\theta = -r(\mu'\alpha' - \mu\alpha)/(\mu' - \mu) = -r\mu\eta\left(\frac{1}{q} - \frac{1}{p}\right)\Big/(\mu'-\mu).$$

We therefore obtain, in terms of p, q, r, which are to be reckoned algebraically as stated above, the formula

$$\mu'\alpha'^2\mathbf{d}q - \mu\alpha^2\mathbf{d}p = -\tfrac{1}{2}\frac{(\mu\eta)^4}{\mu'-\mu}\left(\frac{1}{\mu'^2 q} - \frac{1}{\mu^2 p}\right)\left(\frac{1}{q} - \frac{1}{p}\right)\frac{1}{r}\ldots(\text{A}');$$

the aberrations $\mathbf{d}p$, $\mathbf{d}q$ being positive if onwards with the light (Art. 135).

145. Aberration of a pencil at concentric spherical surfaces.

Let the algebraic values of the radii of the surfaces be $R_1 \ldots R_n$; these will be positive if the surfaces be convex to the incident light, since the centre is the origin. Let the refractive indices of the media be μ_0, $\mu_1 \ldots \mu_n$. Since the centre is the nodal point for all the refractions, the product $\mu_r\eta_r$, where η_r is the radius of the pencil in the rth medium as it is passing the centre, is a constant which we denote by $\mu\eta$.

Hence, for light diverging originally from a point on the axis, we obtain from (A′) the equation

$$\mu_n a_n^2 dq_n = -\tfrac{1}{2}(\mu\eta)^4 \sum_{r=1}^{r=n}\left[\frac{1}{\mu_r-\mu_{r-1}}\left(\frac{1}{\mu_r^2 q_r}-\frac{1}{\mu^2_{r-1}q_{r-1}}\right)\left(\frac{1}{q_r}-\frac{1}{q_{r-1}}\right)\frac{1}{R_r}\right].$$

Also the relation between consecutive geometrical foci is given by the equation

$$\frac{1}{\mu_r q_r}-\frac{1}{\mu_{r-1}q_{r-1}}=\left(\frac{1}{\mu_r}-\frac{1}{\mu_{r-1}}\right)\frac{1}{R_r}.$$

Let $\sum_{r=1}^{r=r}\left(\frac{1}{\mu_r}-\frac{1}{\mu_{r-1}}\right)\frac{1}{R_r}=\frac{K_r}{\mu_0\mu_r}$, so that $\frac{1}{\mu_r q_r}-\frac{1}{\mu_0 p}=\frac{K_r}{\mu_0\mu_r}$; also K_r is the *power* of the refracting system formed by the first r surfaces (cf. iv. Art. 79).

Hence

$$\mu_n a_n^2 dq_n = -\tfrac{1}{2}(\mu\eta)^4 \sum_{r=1}^{r=n}\left[\left\{\frac{1}{\mu_0}\left(\frac{K_r}{\mu_r^2}-\frac{K_{r-1}}{\mu^2_{r-1}}\right)+\left(\frac{1}{\mu_r}-\frac{1}{\mu_{r-1}}\right)\frac{1}{\mu_0 p}\right\}\right.$$
$$\left.\times\left\{\frac{K_r-K_{r-1}}{\mu_0(\mu_r-\mu_{r-1})}+\frac{1}{\mu_0 p}\right\}\frac{1}{R_r}\right].$$

In this summation the coefficient of $(1/\mu_0 p)^2$ is $K_n/\mu_0\mu_n$, while the coefficient of $(1/\mu_0 p)$ is

$$\frac{1}{\mu_0}\sum_{r=1}^{r=n}\left[\left\{\frac{K_r}{\mu_r^2}-\frac{K_{r-1}}{\mu^2_{r-1}}-\frac{K_r-K_{r-1}}{\mu_r\mu_{r-1}}\right\}\frac{1}{R_r}\right],$$

i.e. $$\frac{1}{\mu_0}\sum_{r=1}^{r=n}\left[\left(\frac{K_r}{\mu_r}+\frac{K_{r-1}}{\mu_{r-1}}\right)\left(\frac{1}{\mu_r}-\frac{1}{\mu_{r-1}}\right)\frac{1}{R_r}\right]=\frac{1}{\mu_0^2}\sum_{r=1}^{r=n}\left(\frac{K_r^2}{\mu_r^2}-\frac{K^2_{r-1}}{\mu^2_{r-1}}\right)=\frac{K_n^2}{\mu_0^2\mu_n^2}.$$

The term independent of p does not lend itself to reduction.

Hence we may write

$$\mu_n a_n^2 dq_n = -\tfrac{1}{2}(\mu\eta)^4\left\{\frac{K_n}{\mu_0\mu_n}\left(\frac{1}{\mu_0 p}+\frac{1}{2}\frac{K_n}{\mu_0\mu_n}\right)^2-A\right\},$$

where A is a constant, which may be thrown into the form

$$\frac{1}{4}\left(\frac{K_n}{\mu_0\mu_n}\right)^3-\sum_{r=1}^{r=n}\left\{\frac{K_rK_{r-1}}{\mu_0^2\mu_r\mu_{r-1}}\left(\frac{1}{\mu_r}-\frac{1}{\mu_{r-1}}\right)\frac{1}{R_r}+\frac{\mu_r-\mu_{r-1}}{(\mu_r\mu_{r-1})^2}\left(\frac{1}{R_r}\right)^3\right\}.$$

In the case where $\mu_0=\mu_n=1$ the system gives the same geometrical foci as a thin lens at the centre of focal length $f\,(=1/K_n)$, and we may write

$$dq = -\tfrac{1}{2}p^2q^2a_0^2\left\{\frac{1}{4f}\left(\frac{1}{q}+\frac{1}{p}\right)^2-A\right\},$$

where $1/q-1/p=1/f$.

146. Aberration of a pencil on passing through a thin lens.

In telescopic lenses, but not in microscopic lenses, the axial thickness is, as a rule, practically proportional to the square of the aperture; *e.g.* in a double-convex lens, supposed very thin at its edge, Newton's formula for curvature gives $t=y^2(1/2r+1/2s)$.

Hence in such lenses we may neglect the effect of the thickness on the aberration of all pencils whose angles of divergence are small enough for the approximate formulae of aberration to be applicable, and treat the lens as thin, so that the radius y of the pencil is the same at each refraction.

The index of refraction of the lens with respect to the surrounding medium being μ, a double application of (A) Art. 141 gives

$$\alpha_2^2 dv = \tfrac{1}{2}\left(\frac{\mu}{\mu-1}\right)^2 y\left\{\left(\frac{\alpha_1}{\mu}-\alpha_0\right)(\alpha_1-\alpha_0)^2+\left(\alpha_2-\frac{\alpha_1}{\mu}\right)(\alpha_2-\alpha_1)^2\right\}$$

$$= \tfrac{1}{2}\frac{\mu y}{(\mu-1)^2}(\alpha_2-\alpha_0)\left[(\mu+2)\alpha_1^2-(2\mu+1)\alpha_1(\alpha_2+\alpha_0)\right.$$
$$\left.+\mu(\alpha_2^2+\alpha_2\alpha_0+\alpha_0^2)\right]\ldots\ldots\text{(I)},$$

where the successive angles of divergence α_0, α_1, α_2 are connected with the angular apertures θ, ϕ of the surfaces by the equations

$$\mu\alpha_1-\alpha_0=(\mu-1)\theta,$$
$$\alpha_2-\mu\alpha_1=(1-\mu)\phi,$$
$$\alpha_2-\alpha_0=(\mu-1)(\theta-\phi)=\kappa y.$$

To obtain the aberration produced by successive thin lenses, we may add the successive values of the right-hand side of (I), giving the proper values to the angles of divergence and the radius of the pencil at each lens, which correspond to any given origin of light on the axis.

147. Lens which makes the aberration a minimum.

The position of the origin of light and the power of the lens being given, we may regard α_0 and α_2 as known quantities, and α_1 as varying with the curvatures of the surfaces.

Completing the square in α_1, we write (I) in the form

$$\alpha_2^2 dv = \tfrac{1}{2}\frac{\mu\kappa y^2}{(\mu-1)^2}\left[(\mu+2)\left\{\alpha_1-\frac{2\mu+1}{2(\mu+2)}(\alpha_2+\alpha_0)\right\}^2\right.$$
$$\left.+\frac{(4\mu-1)(\alpha_2^2+\alpha_0^2)-2(2\mu^2+1)\alpha_0\alpha_2}{4(\mu+2)}\right]\ldots\text{(i)}.$$

The second term in this bracket has real factors, and it will be negative if α_2/α_0 lie between the two values

$$[2\mu^2+1\pm 2(\mu-1)\{\mu(\mu+2)\}^{\frac{1}{2}}]/(4\mu-1).$$

It is therefore possible to choose α_1 so that the aberration vanishes for pencils satisfying this condition. But since $\alpha_2/\alpha_0 = 1 + u/f$, the distance u of the origin of light from the lens must lie between the values

$$2f(\mu-1)\,[\mu-1 \pm \{\mu(\mu+2)\}^{\frac{1}{2}}]/(4\mu-1).$$

For any kind of glass these limiting positions are very near the lens; *e.g.* if $\mu = \frac{3}{2}$, the limits are $\frac{1}{10}f(1 \pm \sqrt{21})$.

Excluding positions of the origin of light between these points, we see that the bracket in (i) is positive, and therefore that the aberration has the same sign as the power of the lens. Hence as a rule a single convergent lens or a system of convergent lenses is under-corrected; *i.e.* the extreme rays cut the axis behind the geometrical focus, and the cusp of the caustic points in the direction of the emergent light.

For a given origin of light the aberration will be a minimum when $\alpha_1 = \dfrac{2\mu+1}{2(\mu+2)}(\alpha_2 + \alpha_0)$; from which we deduce

$$\theta = p\alpha_2 + q\alpha_0, \quad \phi = p\alpha_0 + q\alpha_2 \quad \text{.................(ii)}$$

where

$$p = (2\mu^2+\mu)/2(\mu+2)(\mu-1), \quad q = (2\mu^2-\mu-4)/2(\mu+2)(\mu-1).$$

These equations give the angular apertures of the two surfaces of the lens, and the ratio of their radii; the actual values of the radii are to be found from the required focal length.

The most important case is that in which the aberration is a minimum at the second principal focus. This is given by α_0 vanishing, and therefore $\sigma/\rho = \theta/\phi = p/q$.

Since $1/f = (\mu-1)(1/\rho - 1/\sigma)$, we deduce

$$\rho = 2f(\mu-1)(\mu+2)/(2\mu^2+\mu), \quad \sigma = 2f(\mu-1)(\mu+2)/(2\mu^2-\mu-4).$$

If we take $\mu = \frac{3}{2}$, then the radii are $\frac{7}{12}f$ and $-\frac{7}{2}f$. This lens, in which the curvature of the first face is six times the curvature of the second face, is known as a *crossed* lens; it is a double-concave or double-convex lens, according as it is divergent or convergent. But the expression $2\mu^2 - \mu - 4$ in the denominator of σ vanishes when $\mu = \frac{1}{4}(1+\sqrt{33}) = 1{\cdot}686\ldots$, a value attained by the more refractive kinds of glass, as dense flint; and therefore for values of μ near this value, σ is very large, and changes very rapidly as μ is varied.

It therefore follows that a lens with its second face plane gives at its second principal focus aberration very slightly in excess of the minimum aberration, and also that in calculating the aberration at the focus more exactly, the thickness of the lens should be taken into account (cf. Art. 148).

To compare different lenses for aberration, we may take the radius of the least circle of aberration at the focus as a measure of their performance.

The radius is $\frac{1}{4}\alpha_2 dv$, and therefore, putting α_0 and the square in α_1 zero in (i), we obtain for its minimum value

$$\frac{1}{32}\frac{\mu(4\mu-1)}{(\mu-1)^2(\mu+2)}\frac{y^3}{f^2}.$$

For the lens having its second face plane, substitute $\mu\alpha_1=\alpha_2$, $\alpha_0=0$, directly in (I), and we obtain $\frac{1}{8}\frac{\mu^3-2\mu^2+2}{(\mu-1)^2\mu}\frac{y^3}{f^2}$ for the radius.

For an equi-convex or equi-concave lens, we have $\alpha_2=2\mu\alpha_1$ when $\alpha_0=0$, and the radius of the least circle of aberration is

$$\frac{1}{32}\frac{4\mu^3-4\mu^2-\mu+2}{(\mu-1)^2\mu}\frac{y^3}{f^2}.$$

The variations in the value of σ, and the practical equality of the concavo-plane or convexo-plane lens with the lens of minimum aberration are shewn in the following table. The radius of the circle of aberration at the focus is ny^3/f^2.

μ	ρ/f	σ/f	$n=$		
			Minimum	Plane face	Equi-curved
1·50	7/12	−7/2	45/168	49/168	70/168
1·55	·6	−5·2	·235	·245	·37
1·60	9/14	−9	·208	·212	·35
1·65	·67	−24	·187	·188	·33
1·686	·686	∞	·174	·174	·32
1·70	·69	+64	·170	·170	·31
1·75	5/7	+15	·155	·157	·30

Lastly it may be mentioned that the equi-convex lens is best fitted for the practical determination of its focal length by observation of the minimum distance between an object and image.

For in that case $\alpha_2+\alpha_0=0$, and therefore the aberration, which is now equal to $\frac{1}{2}\frac{\mu\kappa y^2}{(\mu-1)^2}\left[(\mu+2)\left(\frac{\alpha_1}{\alpha_0}\right)^2+\mu\right]$, is a minimum

when α_1 is zero. This gives $\theta=-\phi$; the lens is therefore equi-convex.

148. To find the effect of the thickness of a lens on the aberration at the second principal focus.

Let t be the thickness, κ_1 the power of the first surface, K the power of the lens. If a_0 be zero, and y_1 be the radius of the pencil at the first surface, we have

$$\mu a_1=\kappa_1 y_1,$$

$$y_2=L_t\left[a_0\left(u+\frac{t}{\mu}+\kappa_1 u\frac{t}{\mu}\right)\right]=y_1\left(1+\kappa_1\frac{t}{\mu}\right),$$

$$a_2=L_t\left[a_0\left(uK+\frac{\partial K}{\partial \kappa_1}\right)\right]=y_1 K.$$

Hence applying (A) twice, the aberration at the second principal focus is given by the equation

$$y_1^2K^2 dv=\frac{1}{2}\left(\frac{\mu}{\mu-1}\right)^2 y_1^4\left[\frac{\kappa_1}{\mu^2}\left(\frac{\kappa_1}{\mu}\right)^2+\left(1+\kappa_1\frac{t}{\mu}\right)\left(K-\frac{\kappa_1}{\mu^2}\right)\left(K-\frac{\kappa_1}{\mu}\right)^2\right],$$

i.e.

$$dv=\frac{1}{2}\frac{(y_1/K)^2}{\mu^2(\mu-1)^2}\left[\{\mu^4K^3-(2\mu^3+\mu^2)K^2\kappa_1+(\mu^2+2\mu)K\kappa_1^2\}\left(1+\frac{\kappa_1 t}{\mu}\right)-\kappa_1^4\frac{t}{\mu}\right].$$

If this be made a minimum by variation of κ_1, then

$$-(2\mu^3+\mu^2)K^2+2(\mu^2+2\mu)K\kappa_1+\frac{t}{\mu}\{\mu^4K^3-2(2\mu^3+\mu^2)K^2\kappa_1 +3(\mu^2+2\mu)K\kappa_1^2-4\kappa_1^3\}=0.$$

Now supposing the thickness of the lens small in comparison with its focal length, the first approximation to the solution of this equation is obtained by neglecting the terms multiplying t, and is therefore $\kappa_1=\dfrac{2\mu^2+\mu}{2(\mu+2)}K$, as before.

On substituting this value in the coefficient of t, we obtain as the second approximation

$$\kappa_1=\frac{(\mu-1)}{\rho}=\frac{2\mu^2+\mu}{2(\mu+2)}\frac{1}{f}+\frac{3}{8}\frac{4\mu^4+3\mu^3+2\mu}{(\mu+2)^4}\frac{t}{f^2},$$

and then solving for κ_2 from the equation $K=\kappa_1+\kappa_2+\kappa_1\kappa_2 t/\mu$ we have

$$\kappa_2=\frac{1-\mu}{\sigma}=\frac{4+\mu-2\mu^2}{2(\mu+2)f}+\frac{8\mu^5+20\mu^4+5\mu^3-80\mu^2-110\mu-32}{8(\mu+2)^4}\frac{t}{f^2}.$$

For $\mu=3/2$ these values are

$$\rho=\frac{7f}{12}\left(1-\frac{267}{2744}\frac{t}{f}\right),$$

$$\sigma=-\frac{7f}{2}\left(1+\frac{1585}{1372}\frac{t}{f}\right);$$

while for $\mu=\frac{1}{4}(1+\sqrt{33})$, $(2\mu^2-\mu-4=0)$;

$$\rho=(\mu-1)f\left[1-\frac{3}{8}\frac{t}{f}\frac{17\frac{1}{4}\mu+21}{(\mu+2)^4}\right]=(\mu-1)f[1-(\cdot 1018)\,t/f],$$

$$\sigma=(6\cdot 74)f^2/t.$$

149. Aberration in Eye-pieces.

The formula for aberration in a thin lens may also be employed to discuss the best forms for the lenses in an eye-piece.

When an eye-piece is used in an astronomical telescope normally adjusted, the rays from a star on the axis emerge to a first approximation parallel to the axis of the telescope. Hence the aberration formula will give the final angle made by the extreme rays with the axis (cf. iii. Art. 143).

Let κ_1, κ_2 be the powers of the field-glass and eye-glass respectively; and let the powers of their surfaces be in the former $\frac{1}{2}\kappa_1(1+x_1)$ and $\frac{1}{2}\kappa_1(1-x_1)$, and in the latter $\frac{1}{2}\kappa_2(1+x_2)$ and $\frac{1}{2}\kappa_2(1-x_2)$. Let the distance between the lenses be a, the power of the eye-piece κ. Let α_0, α_1, α_2, α_3, α_4 be the successive angles of divergence, y_1, y_2 the radii of the pencil as it passes the two lenses.

Then in applying Cotes' formula to the path of the ray reversed, α_4 is zero, but $v\alpha_4$ is finite and equal to y_2.

Taking $\mu = \frac{3}{2}$ for simplicity, the final angle $\mathbf{d}\alpha_4$ is given by a repeated application of (I) Art. 146, and we have

$$y_2 \mathbf{d}\alpha_4 = \tfrac{3}{2}\kappa_1 y_1^2 \{7\alpha_1^2 - 8\alpha_1(\alpha_2+\alpha_0) + 3(\alpha_2^2 + \alpha_2\alpha_0 + \alpha_0^2)\}$$
$$+ \tfrac{3}{2}\kappa_2 y_2^2 \{7\alpha_3^2 - 8\alpha_3(\alpha_4+\alpha_2) + 3(\alpha_4^2 + \alpha_4\alpha_2 + \alpha_2^2)\}.$$

Also
$$\tfrac{3}{2}\alpha_3 = \tfrac{1}{2}\kappa_2(1-x_2)\,y_2,$$
$$\alpha_2 = \kappa_2 y_2,$$
$$\tfrac{3}{2}\alpha_1 = L_t\,[\alpha_4\{1+\kappa_2 v + \tfrac{1}{2}\kappa_1(1-x_1)(v+a+\kappa_2 va)\}]$$
$$= \{\kappa_2 + \tfrac{1}{2}\kappa_1(1-x_1)(1+\kappa_2 a)\}\,y_2$$
$$= \tfrac{1}{2}\{(1+x_1)\kappa_2 + (1-x_1)\kappa\}\,y_2,$$
$$\alpha_0 = \kappa y_2,$$

and
$$y_1 = L_t\,[\alpha_4\{v+a+\kappa_2 va\}] = y_2(1+\kappa_2 a).$$

If these values be substituted above we obtain

$$\mathbf{d}\alpha_2 = \tfrac{1}{6}y_2^3\kappa_1(1+\kappa_2 a)^2\{(10-10x_1+7x_1^2)\kappa_2^2 + (10+10x_1+7x_1^2)\kappa^2$$
$$-(7+14x_1^2)\kappa_2\kappa\}$$
$$+\tfrac{1}{6}y_2^3\kappa_2^3(10+10x_2+7x_2^2).$$

For instance, in Ramsden's eye-piece $\kappa_1 = \kappa_2 = \frac{2}{3}\kappa$, and $a\kappa_2 = -\frac{2}{3}$, whence

$$\mathbf{d}\alpha_4 = \frac{(y_2\,\kappa)^3}{2^7 . 3^2}\{166 + 70x_1 + 7x_1^2 + 81\,(10 + 10x_2 + 7x_2^2)\}$$

$$= \frac{\alpha_0^3}{2^7 . 3^2}\{511\tfrac{5}{7} + 7\,(x_1 + 5)^2 + 567\,(x_2 + \tfrac{5}{7})^2\}.$$

The angle made by the extreme emergent rays with the axis will therefore be least if $x_1 = -5$ and $x_2 = -5/7$. Hence the curvatures of the two faces of the field-glass being in the ratio $1 + x_1 : x_1 - 1$, or $2 : 3$, we have $\rho = -\frac{1}{4}f$, $\sigma = -\frac{1}{6}f$ for the field-glass; and the eye-glass must be a *crossed* lens in which $\rho = \frac{7}{2}f$, $\sigma = -\frac{7}{12}f$, the flatter face being towards the incident light. The value of α_0 is the ratio of the radius of the object-glass to its focal length.

If we take α_0 as $1/30$ the minimum value of $\mathbf{d}\alpha_4$ is about $3''\cdot 4$; if however the lenses are plano-convex and convexo-plane, as is usual in practice, we must put $x_1 = -1$, $x_2 = +1$, and the value will be about $15''$, an angle even so too small to be visible.

It should be noticed that this investigation applies solely to rays from a point on the axis, or the centre of the field of view; and that nothing is determined about the shapes of the lenses which give the field of view most free from extra-axial distortion.

150. Aplanatic Object-glasses.

Although it is not possible to make the aberration zero at the second principal focus of a single lens, yet this end can be attained by a combination of two lenses.

The aberration produced by a lens of given focal length in a pencil diverging from a given point contains one indeterminate quantity. Hence in an achromatic object-glass, which is composed as a rule of a convergent crown-glass lens placed in front of a divergent flint-glass lens, the final aberration will involve two indeterminate quantities.

We may therefore choose some relation between the shapes of the lenses, provided it does not involve their focal lengths, which are already connected by the condition of achromatism.

Now in object-glasses of diameter not greater than four inches, it is usual to cement the lenses together, which prevents some loss of light by internal reflections, but in larger objectives the danger of fracture of the lenses, owing to the unequal expansions of crown and flint-glass, forbids this course.

It is the practice in these objectives to separate the lenses slightly; by which means both the chromatic and the spherical aberrations are more perfectly corrected. The formulae for the aberration become extremely complicated, when the distance apart and the thicknesses of the lenses are taken into account; practically they are not used, but the paths of rays, incident in different zones of the lens, are calculated by solution of the triangles they form with the axis.

We therefore confine ourselves to a telescopic objective of small aperture, and disregard the thickness, which is after all very small compared with the focal length.

For this purpose it is convenient to rewrite the formula (I) Art. 146 for aberration in a thin lens in the forms

$$\alpha_2^2 \mathrm{d}v - \alpha_0^2 \mathrm{d}u = \tfrac{1}{2}\kappa y^2 \{a\theta^2 - (b\alpha_0 + c\delta)\,\theta + d\alpha_0^2 + e\alpha_0\delta + f\delta^2\} \;\ldots\; \text{(i)}$$

$$= \tfrac{1}{2}\kappa y^2 \{a\phi^2 - (b\alpha_2 - c\delta)\,\phi + d\alpha_2^2 - e\alpha_2\delta + f\delta^2\} \ldots \text{(ii)},$$

where $a = (\mu + 2)/\mu$, $b = 4(\mu + 1)/\mu$, $c = (2\mu + 1)/(\mu - 1)$,

$d = (3\mu + 2)/\mu$, $e = (3\mu + 1)/(\mu - 1)$, $f = \mu^2/(\mu - 1)^2$,

and θ, ϕ, α_0, α_2 have the same meanings as in that article, while δ is the deviation produced by the lens.

Hence in the case of two thin lenses of different materials, placed in contact with each other,

$$\alpha_4^2 \mathrm{d}v = \tfrac{1}{2}\kappa y^2 \{a\phi^2 - (b\alpha_2 - c\delta)\,\phi + d\alpha_2^2 - e\alpha_2\delta + f\delta^2\}$$

$$+ \tfrac{1}{2}\kappa' y^2 \{a'\theta'^2 - (b'\alpha_2 + c'\delta')\,\theta' + d'\alpha_2^2 + e'\alpha_2\delta' + f'\delta'^2\} \ldots \text{(iii)},$$

where δ, δ' are the deviations produced by the lenses respectively, ϕ is the angular aperture of the second surface of the first lens, θ' that of the first surface of the second lens, and α_2 is the angle of divergence of the ray as it passes from the first to the second lens.

The powers κ, κ' of the two lenses are connected by the condition of achromatism $\varpi\kappa + \varpi'\kappa' = 0$.

If the incident light be parallel to the axis, then $a_2 = \delta$; and since, on equating the aberration to zero, we obtain an equation which only involves ratios, we may suppose the compound lens to be convergent, and to produce a deviation represented by unity.

We must then substitute,

$$a_2 = \delta = -\varpi'/(\varpi' - \varpi), \text{ and } \delta' = \varpi/(\varpi' - \varpi);$$

since these values satisfy

$$\delta + \delta' = -1 \text{ and } \varpi\delta + \varpi'\delta' = 0.$$

The numerical values of the coefficients a, a', &c. for the mean ray adopted as standard must be substituted. We may replace $(d - e + f)$ by its value

$$(\mu^3 - 2\mu^2 + 2)/\mu(\mu - 1)^2 \text{ or } 1 + (2 - \mu)/\mu(\mu - 1)^2.$$

If the two faces be cemented together $\theta' = \phi$, and we then obtain a quadratic equation for ϕ; the smaller root of this equation must be taken, as that indicates shallower curves, and less outstanding aberrations of the higher orders.

The relative apertures of the other surfaces are then given by

$$\theta = \phi + \delta/(\mu - 1), \quad \phi' = \phi - \delta'/(\mu' - 1);$$

and finally, if f be the actual focal length of the compound lens, the radii of curvature of the surfaces in order are f/θ, f/ϕ, and f/ϕ'.

With the following pairs of values of μ, μ' and the corresponding values of the ratio of the dispersive powers ϖ/ϖ' (cf. Art. 123), I find that the relative apertures are as in the following table.

μ	μ'	ϖ/ϖ'	θ	ϕ	ϕ'
1·5180	1·7191	·532	−2·265	1·861	+·280
1·5203	1·6576	·563	−2·304	2·097	+·135
1·5179	1·6202	·584	−2·318	2·323	−·094

Although the coefficients of the quadratic equation in ϕ vary considerably with the ratio ϖ/ϖ', the value of θ in any case does not differ much from 2·3, and the corresponding values of ϕ, ϕ' may be calculated from the known values of δ and δ' for each value of ϖ/ϖ'. Practically the form of the best objectives is very simple; the crown-glass lens is double-convex of equal curvatures, while the flint-glass lens is placed very nearly in contact, and the curvature

of its second face calculated by the condition of achromatism. This face is very nearly plane, concave however if ϕ' be negative. For the third set of values of μ, μ' above, this method would give

$$-\theta = \phi = 2{\cdot}3203, \quad \phi' = {\cdot}054,$$

and the radius of the least circle of aberration at the second focus is ${\cdot}05\,(y/f)^2\,y$, a quantity quite inappreciable.

151. Sir John Herschel suggested that instead of cementing the surfaces the aberration should be made zero not only for rays parallel to the axis, but also for rays from a point at a considerable distance. This might be useful in objectives for viewing terrestrial objects.

To this end we must substitute above (iii. Art. 150) $\alpha_2 = \delta + \alpha_0$, and equate to zero not only the term independent of α_0 but also the coefficient of α_0. This gives a linear relation

$$\kappa\,\{b\phi - (2d - e)\,\delta\} + \kappa'\,\{b'\theta' - (2d' + e')\,\delta'\} = 0,$$

between ϕ and θ'.

There is then a very small residual aberration,

$$\mathbf{d}v = \tfrac{1}{2}\,(\kappa d + \kappa' d')\,y^2\,(\alpha_0/\alpha_4)^2.$$

I find that the corresponding values of θ, ϕ, θ', ϕ' are

μ	μ'	ϖ/ϖ'	θ	ϕ	θ'	ϕ'
1·5180	1·7191	·532	−1·506	2·622	2·566	+·917
1·5203	1·6576	·563	−1·500	2·898	2·841	+·882
1·5179	1·6202	·584	−1·500	3·142	3·095	+·832

The values of ϕ, θ' in each case are nearly equal, so that the two lenses are really very close at all points; and these values of ϕ are very nearly the greater roots of the quadratic used in Art. 150; also the value of θ is nearly constant at 1·5, from which ϕ might be calculated by the value of δ, θ' from ϕ by the linear equation above, and ϕ' from θ' by the value of δ'.

There is a serious defect attaching to all these corrections for aberration. The formulae involve the refractive index μ; hence if the aberration be made zero at the focus of a compound object-glass for rays of a definite colour, there will be small residual

aberration for rays of other colours. This is known as the chromatic difference of spherical aberrations, and is practically overcome by separating the lenses slightly and also by working, if necessary, the various zones of the object-glass to slightly different curvatures. "A truly spherical object-glass is the exception and not the rule."

152. Indistinctness or Coma.

Nothing in this chapter applies to pencils of light whose origin is off the axis of a refracting system. When the pencil is so small that the squares of the inclinations of the rays to each other may be rejected, but the distance of the origin from the axis is such that squares of the inclinations of the rays to the axis and of the angles of incidence must be retained, the final pencil is determined by passing through two small focal lines, and the section of the pencil by a plane perpendicular to the axial ray or by one perpendicular to the axis of the system is a quartic curve in place of a circle. There is nothing which can definitely be taken as the image of the origin of light, in the same way as we take the geometrical focus for the first approximation, or the least circle of aberration for the second. The indistinctness of the image arising from this cause, even in a system corrected for axial aberrations, is known as Coma. The analytical expressions for this indistinctness and also for distortion are investigated below, Chap. XIV.

EXAMPLES.

1. A pencil of rays diverging from a point passes through any number of parallel plates, having air as the first and last medium. Shew that the aberration is independent of the position of the origin of light, and of the order of the plates; and that its value is

$$\Sigma \left\{ \frac{1}{\mu_r} - \frac{\cos\phi}{(\mu_r^2 - \sin^2\phi)^{\frac{1}{2}}} \right\} t_r,$$

where t_r is the thickness of the medium of index μ_r, and ϕ is the angle of incidence on the first surface.

2. A pencil of rays passes through any number of media of thicknesses $t_1, t_2 \dots t_{n-1}$ and of refractive indices $\mu_1, \mu_2 \dots \mu_{n-1}$, separated by parallel planes. If y be the radius of the pencil at incidence on the first surface, u the distance of the origin of light from that surface, and μ_0, μ_n the indices of the initial and final media, the radius of the least circle of aberration of the emergent pencil is

$$\frac{1}{8}\frac{\mu_0^3 y^3}{u^3}\left\{\frac{u}{\mu_0}\left(\frac{1}{\mu_0^2} - \frac{1}{\mu_n^2}\right) + \sum_{r=1}^{r=n}\frac{t_r}{\mu_r}\left(\frac{1}{\mu_r^2} - \frac{1}{\mu_n^2}\right)\right\}.$$

3. An eye, the radius of whose pupil is a small quantity r, views directly a point at distance c through a plate of thickness t and refractive index μ. Shew that the rays which enter the eye diverge as from a small circle of radius approximately equal to $\frac{1}{8}\frac{(\mu^2-1)\,r^3t}{\{\mu c-(\mu-1)\,t\}^3}$.

4. A ray from a point Q on the axis OA of a spherical mirror is reflected at R, and the tangent plane to the mirror at R cuts the axis in T. Shew that the aberration is accurately $\frac{1}{2}AT.OQ^2/FQ.F'Q$; where F is the principal focus of the mirror, and F' the point in which a ray, incident at R parallel to the axis, cuts the axis after reflection.

5. The radius of the least circle of aberration of a pencil after reflection at a spherical surface is $\frac{1}{4}\frac{q^2y^3}{vr^3}$, where q and v are the distances of the geometrical focus from the centre and vertex respectively.

6. A pencil of rays parallel to the axis is incident on Cassegrain's telescope, the radius of the pencil being y. Shew that the radius of the least circle of aberration formed after reflection at the two mirrors is $\frac{1}{32}y^3(Ff^3-a^2x^2)/xF^3f^2$, where F and f are the focal lengths of the two mirrors, x and a are the distances of the focus of the small mirror and of the circle of aberration respectively from the focus of the large mirror.

7. A pencil is refracted directly at a spherical surface from a medium of index μ into one of index μ'; shew that, if η be the radius of the incident pencil as it passes the centre, the radius of the least circle of aberration is with the usual notation $\frac{1}{8}\frac{(\mu'-\mu)\,\mu}{\mu'^2}\frac{\eta^3}{r^2}\left(\frac{r}{p}\sim\frac{q}{r}\right)$.

8. A pencil diverging from a point is refracted directly through a sphere of refractive index μ. Shew that, if η be the radius of the pencil in the sphere when passing the centre, and f be the focal length of the sphere, the approximate formula connecting the distances of conjugate foci from the centre is

$$\frac{1}{q}-\frac{1}{p}=\frac{1}{f}+\frac{\mu^2\eta^2}{8f}\left\{\frac{\mu}{(\mu-1)^2}\frac{1}{f^2}-\left(\frac{1}{p}+\frac{1}{q}\right)^2\right\}.$$

9. A pencil of rays is refracted directly through a hemisphere; the distance of the origin of light from the plane surface, which is that of first incidence, being u, shew that the aberration of a ray incident at a distance y from the axis is

$$\frac{(\mu-1)\,y^2}{2r}\,\frac{\mu u\,(\mu u+r)\,(\mu^2u-r)+(\mu+1)\,r^3}{u\,\{\mu\,(\mu-1)\,u-r\}^2}.$$

10. The thickness of a concavo-convex meniscus is $\frac{\mu+1}{\mu}(\sigma-\rho)$, where ρ and σ are the radii of its surfaces. Shew that the unit points are absolutely aplanatic.

11. A pencil of parallel rays is refracted through a thin lens; if the lens be turned round so that the order of the surfaces is reversed, the aberration is in the same direction as before and differs from its previous value by

$\frac{1}{2}y^2\left(1+\frac{1}{\mu}\right)\left(\frac{1}{\rho}+\frac{1}{\sigma}\right)$. Shew that in the general case the aberration is increased by $\frac{1}{2}y^2\left(1+\frac{1}{\mu}\right)\left(\frac{1}{\rho}+\frac{1}{\sigma}\right)(1-m^2)$, when m is the linear magnification for the origin of light.

12. Shew that, if v', v'' be the distances from a thin lens of the two positions of the geometrical focus for which the aberration vanishes, u' and u'' the corresponding distances of the origin of light, the aberration at a geometrical focus at distance v from the lens is

$$\frac{3\mu+2}{2\mu}\frac{(v-v')(v-v'')}{v'v''}\frac{y^2}{f} \text{ or } \frac{3\mu+2}{2\mu}\frac{v^2}{u^2}\frac{(u-u')(u-u'')}{u'u''}\frac{y^2}{f}.$$

Prove that if these points be real, the ratio of the curvatures of the surfaces of the lens must lie between $\{\sqrt{(3\mu+2)\mu}+1\}:\{\sqrt{(3\mu+2)\mu}-1\}$, and its reciprocal.

13. Shew that for a thin lens of given focal length f the greatest possible distance from the lens of a geometrical focus at which the aberration vanishes is $2(\mu-1)\{\sqrt{\mu(\mu+2)}+\mu-1\}f/(4\mu-1)$, and that then the curvature of the first surface is to that of the second as

$$2(\mu+1)\sqrt{\mu}+\sqrt{\mu+2} \;:\; 2(\mu+1)\sqrt{\mu}-\sqrt{\mu+2}.$$

14. A pencil diverging from a point at distance $2f$ in front of the first lens passes through a system of n thin equi-convex lenses, each of focal length f and at a distance $4f$ apart. Shew that the radius of the least circle of aberration of the emergent pencil is $\frac{n}{16}\left(\frac{\mu}{\mu-1}\right)^2\frac{y^3}{f^2}$, where y is the radius of aperture of the first lens.

15. A ray from the first principal focus of a thick lens with one plane face makes a small angle a_0 with the axis. Shew that the angle made with the axis by the emergent ray is approximately

$$\tfrac{1}{2}a_0^3\left\{\frac{\mu^3-2\mu^2+2}{\mu(\mu-1)^2}+\frac{\mu^2-1}{\mu^3}\frac{t}{f}\right\} \text{ or } \tfrac{1}{2}a_0^3\left(\frac{\mu}{\mu-1}\right)^2,$$

according as the light is incident first on the plane or the curved surface of the lens.

CHAPTER IX.

ILLUMINATION.

153. A LUMINOUS body is a source of energy in various ways; the heat rays, the chemical rays, and the visible rays all possess energy. The relative distribution of the energy varies with the nature of the source, but as far as regards the apparent brightness of the source and the illumination it produces on a surface, we are only concerned with the visible rays; and all methods for comparing the intensities of two sources of light practically resolve into an appeal to the sensations of the retina. The eye can judge with fair accuracy whether two similar surfaces appear equally bright when illuminated by light from two monochromatic sources; it cannot judge with the same accuracy between lights of different colours, nor can we estimate directly the ratio of unequal illuminations.

In the methods described in treatises on Photometry*, the comparison is made between two sources of light, which for practical purposes are treated as concentrated at points; and the rays leave the sources in certain directions, definite for the purpose of experimental comparison; then two sources are considered of equal intensity if they make two similar areas, placed so that the light falls on them in exactly the same manner in both cases, appear equally bright. In fact, comparison of illuminations is substituted for comparison of intensities.

The comparison of unequal intensities is effected by arranging the positions of the sources with regard to the illuminated areas so that these still appear equally bright. The numerical relation between the intensities is deduced from the following considerations.

* Palaz, *Photométrie*, Paris, 1892. Dredge, *Electric Illumination*, London, 1885.

Since in a homogeneous medium light is transmitted in straight lines, the same amount of light from any origin falls on all sections of a cone having the origin as vertex. Hence, considering a cone of small solid angle, so that the intensity of emission may be considered constant within this angle, we see that the illumination *per unit area* at any point varies inversely as the square of the distance of that point from the source of light; and further we see that, when the light is incident obliquely at angle ϕ on any small plane section of this cone, the illumination *per unit area* varies as $\cos\phi$; since the area of an oblique section is $\sec\phi$ times the area of a right section at the same point.

If then two sources of light render two similar small areas at distances r_1 and r_2 equally bright, their intensities C_1 and C_2 are connected by the equation $C_1\dfrac{\cos\phi_1}{r_1^2} = C_2\dfrac{\cos\phi_2}{r_2^2}$, where ϕ_1 and ϕ_2 are the angles of incidence of the rays.

The unit of intensity is that of a definite source of light, as a candle of known material and make, in England, or a standard oil-lamp, the Carcel lamp, in France.

To measure amount of light we take as unit the light which is incident normally on unit area at unit distance from the standard source of light. The total amount of light sent out in all directions by a source of candle-power C will be $4\pi C$, if the source emit light uniformly in all directions. The light from such a source, which falls on a surface subtending a solid angle Ω at the source, will be $C\Omega$; and this is the measure of the illumination of the surface. No source of light, however, is equally powerful in all directions; and the intensity of emission of the standard light, or of any other light, as the electric arc, must be determined experimentally for each azimuth.

Passing to the case where the illumination of the area considered is produced by an extended bright surface, such as a glowing substance or any surface in diffused sunlight or a piece of roughened glass in front of a source of light, we define the *intrinsic brightness* I of the surface as a quantity such that IdS is the amount of light sent out *normally* by the element dS; *i.e.* this element would produce on a small area on the normal to it the same illumination as so many unit candles occupying its place.

We should then find that the intensity of the light which leaves the element in a direction making an angle θ with the normal is $IdS\cos\theta$.

This is proved by the fundamental experiment:—all surfaces subtending the same visual angle at any point and of equal intrinsic brightness appear equally bright to the eye placed at that point, and produce equal illuminations on a small area at that point.

Thus an equally bright globe and circular disc are indistinguishable; and the different parts of a bright surface appear equally bright, whatever their inclination to the line of sight.

The area of a right section of the small visual cone being dS, that of an oblique section at the same point will be $dS\sec\theta$, where θ is the angle between the rays emitted from that area and its normal. Hence, the amount of light emitted by the two areas being equal, the amount emitted per unit area in the oblique direction must be $I\cos\theta$.

It follows that the amount emitted by the area dS within a cone of revolution about the normal of semi-vertical angle α is $\pi IdS\sin^2\alpha$; since, the rays being distributed according to this law of emission within each element $d\omega$ ($=\sin\theta d\theta d\psi$) of solid angle, the value of the integral $IdS\int_0^{2\pi}\int_0^{\alpha}\cos\theta\sin\theta d\theta d\psi$ is $\pi IdS\sin^2\alpha$. The total light emitted by the bright area on one side is πIdS.

Combining this law of emission with the previous results, it follows that the amount of light that falls on a small area dS' from a small luminous area dS of intrinsic brightness I is

$$I\frac{\cos\theta\cos\phi}{r^2}dSdS',$$

where θ and ϕ are the angles of emission and incidence, and r is the distance between the elements; this will be taken as the measure of the illumination of the small area dS'.

This expression is symmetrical with regard to the two elements, and may also be written in the forms $I\cos\phi d\omega dS'$ and $I\cos\theta d\omega' dS$, where $d\omega$ is the solid angle subtended by dS at dS', and $d\omega'$ that subtended by dS' at dS.

If the area dS which emits light be in a medium of refractive index μ, then the above expression for the illumination must be

multiplied by μ^2; the energy of the radiation of a body into a medium, both for heat and light, being proportional to the square of the refractive index*.

154. Intensity of Illumination.

The *intensity of illumination* at any point of a surface is the illumination per unit area produced by an extended bright surface on a small element dS' of the surface containing that point.

If I be the intrinsic brightness of the surface, and ϕ the angle of incidence, we have just seen that the intensity is equal to $\iint I \cos\phi\, d\omega$.

Reverting however to the fundamental experiment, a surface of uniform brightness produces the same intensity of illumination as any other surface of equal brightness, which is included within the same visual angle. Draw therefore the tangent cone to the bright surface from the point where the intensity is required; this cone will meet a sphere of unit radius with its centre at that point in a certain curve. The part of the sphere within this curve may be taken as an equivalent bright surface, and the light from it will be emitted normally; therefore the intensity of illumination on a small area at the centre is $I\iint \cos\phi\, d\omega$, and is to be found in any case by evaluating the area of the projection of the spherical curve on the plane of the element.

Since the coefficient of I is a numerical quantity, this will be the intensity of illumination produced by so many candles placed at unit distance on the normal to the small area.

The projection of the spherical area will, as a rule, be most simply obtained by projecting plane areas drawn to form with the spherical area a closed surface. Moreover, since the cosine of the angle of incidence is a linear function of the direction-cosines of the normal to the small area, the intensity of illumination at the origin on the plane $lx + my + nz = 0$ may be written in the form $lI_x + mI_y + nI_z$, where I_x, I_y, I_z are the intensities of illumination on the coordinate planes at the origin, it being supposed that the same parts of the luminous surface illuminate all four planes.

* Clausius. On the concentration of Calorific and Luminous Rays. *Pogg. Ann.* CXXI. 1864.

155. Example.

A spherical luminary produces the same illumination on any area as an origin of light at its centre.

From any point of the area draw the tangent cone to the bright sphere; let a be the semi-vertical angle of this cone, and ϕ the angle between its axis and the normal to the area.

This cone cuts a unit sphere with its centre at the point in a spherical cap of angular radius a, and the projection of this spherical area on any plane is the same as that of the plane circle bounding it.

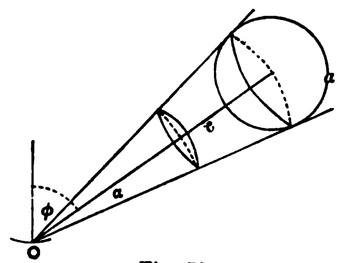


Fig. 79.

Hence the intensity of illumination at O is $I \sin^2 a \cos\phi$, which is equal to $\dfrac{I a^2 \cos\phi}{c^2}$, where a is the radius of the sphere, and c the distance of its centre. The sphere therefore produces the same illumination as a source of light of intensity Ia^2 at its centre; and the total illumination of any surface, every point of which is in full view of the sphere, is $Ia^2 \iint \dfrac{\cos\phi\, dS'}{c^2}$, which is equal to Ia^2 (solid angle subtended by the surface at the centre of the sphere).

156. Example.

To find the illumination produced by a rectangular window, supposed uniformly bright.

Let $ABCD$ be the window, and let the area on which we seek the

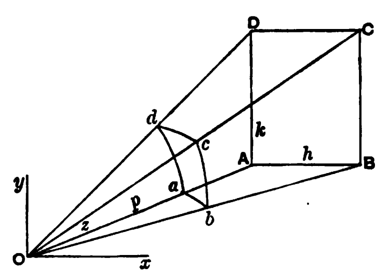


Fig. 80.

illumination be placed at O, a point on the normal to the plane $ABCD$ through the corner A. If it lie at any other point O', draw $O'A'$ perpendicular to the plane of the window, and through A' draw parallels to the sides. Then A' is the corner of four rectangles, and the illumination due to the rectangle AC is the sum of those due to these four rectangles, taken with their proper signs as A' falls inside or outside the window.

The planes through O and the boundary of $ABCD$ cut a sphere, centre O, in great circles, and the equivalent area is the spherical quadrilateral $abcd$. Let the sides AB, AD of the rectangle be h, k, and let $OA=p$. Let the axes of x, y be parallel to the sides, and let OA be the axis of z.

The intensity of illumination at O on the plane $AOD=I_x$

$$=I(\text{projection of } abcd \text{ on that plane})$$

$$=I(\text{area } aOd - \text{projection of area } bOc)$$

$$=\tfrac{1}{2}I(AOD-BOC\cos AOB)$$

$$=\tfrac{1}{2}I[\tan^{-1}(k/p)-\{\tan^{-1}k/(p^2+h^2)^{\frac{1}{2}}\}\,p/(p^2+h^2)^{\frac{1}{2}}] \quad\ldots\ldots\ldots\ldots \text{(i)}.$$

The intensity of illumination at O on the plane $AOB=I_y$

$$=I(\text{area } aOb - \text{projection of area } cOd)$$

$$=\tfrac{1}{2}I(AOB-COD\cos AOD)$$

$$=\tfrac{1}{2}I[\tan^{-1}(h/p)-\{\tan^{-1}h/(p^2+k^2)^{\frac{1}{2}}\}\,p/(p^2+k^2)^{\frac{1}{2}}] \quad\ldots\ldots\ldots\ldots \text{(ii)}.$$

The intensity of illumination at O on the plane $xOy=I_z$

$$=I(\text{projection of } bOc + \text{projection of } cOd)$$

$$=\tfrac{1}{2}I(BOC\cos ABO+COD\cos ADO)$$

$$=\tfrac{1}{2}I[\{\tan^{-1}k/(p^2+h^2)^{\frac{1}{2}}\}\,h/(p^2+h^2)^{\frac{1}{2}}+\{\tan^{-1}h/(p^2+k^2)^{\frac{1}{2}}\}\,k/(p^2+k^2)^{\frac{1}{2}}] \ldots \text{(iii)}.$$

The intensity of illumination at O on an area in full view of the window, and making angles α, β, γ with these planes, will be $I_x\cos\alpha+I_y\cos\beta+I_z\cos\gamma$.

157. Mutual Illumination of Two Surfaces.

It is clear from the fundamental experiment that the mutual illumination of two surfaces, such that every point of one is visible from every point of the other, depends only on their boundaries and not on the surfaces themselves. In other words

$$\iiiint \frac{\cos\theta\cos\phi}{r^2}\,dS\,dS'$$

can be expressed as a double line-integral round their boundaries.

For this purpose we apply Stokes' theorem*:—

$$\int(udx+vdy+wdz)=2\iint(l\xi+m\eta+n\zeta)\,dS,$$

where $\quad 2\xi=\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z},\quad 2\eta=\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x},\quad 2\zeta=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y},$

and l, m, n are the direction-cosines of the normal to the element of surface dS, the line-integral being taken round any closed curve, and the surface-integral over any surface bounded by that curve.

The mutual illumination may be shewn to be equal to

$$\tfrac{1}{2}I\iint\log r\,(dxdx'+dydy'+dzdz').$$

* Lamb's *Hydrodynamics*, p. 37.

For consider first the coefficient of dx' in this integral; if we put $u = \frac{1}{2}\log r$, $v = 0$, $w = 0$, we obtain

$$2\xi = 0,\ 2\eta = (z - z')/2r^2,\ 2\zeta = -(y - y')/2r^2;$$

so that $$\tfrac{1}{2}\int \log r\, dx = \iint \frac{\{m(z - z') - n(y - y')\}}{2r^2} dS \quad\text{.........(i).}$$

Treating the coefficients of dy' and dz' in the same way, we find that the double line-integral above is equal to

$$\iiint \begin{vmatrix} dx' & dy' & dz' \\ l & m & n \\ x - x' & y - y' & z - z' \end{vmatrix} \frac{dS}{2r^2} \quad\text{............ (ii).}$$

The line-integral round the second boundary, which is the coefficient of ldS in (ii), is

$$\int \frac{(z' - z)\,dy' - (y' - y)\,dz'}{2r^2}.$$

If we put $u' = 0$, $v' = (z' - z)/2r^2$, $w' = -(y' - y)/2r^2$, we obtain $2\xi' = -(x' - x)^2/r^4$, $2\eta' = -(y' - y)(x' - x)/r^4$, $2\zeta' = -(z' - z)(x' - x)/r^4$, and the coefficient of ldS is therefore equal to

$$\iint (x' - x)\{l'(x - x') + m'(y - y') + n'(z - z')\} \frac{dS'}{r^4} \quad\text{... (iii).}$$

Treating the coefficients of m and n in the same way, the double line-integral is equal to the double surface-integral

$$\iiiint \{l(x' - x) + m(y' - y) + n(z' - z)\}\{l'(x - x') + m'(y - y') + n'(z - z')\} \frac{dSdS'}{r^4}$$

i.e. to $$\iiiint \frac{\cos\theta\cos\phi}{r^2} dSdS' \quad\text{.....................(iv).}$$

The mutual illumination of two surfaces of intrinsic brightness I may therefore be written as

$$\tfrac{1}{2} I \iint \log r \cos\epsilon\, ds ds',$$

where ϵ is the angle between the tangents to the boundaries.

From the proof of Stokes' theorem, it is plain that the directions of integration round the two boundaries must bear the same screw relation to the normals to the two surfaces, which are to be drawn so that the angles θ, ϕ are always acute.

From equation (ii) above, it is clear that the intensity of illumination produced at the point (x, y, z) on a plane of direction-cosines (l, m, n) by a uniformly bright surface S' may be written as

$$lI_x + mI_y + nI_z,$$

where $$I_x = I\int\{(z' - z)\,dy' - (y' - y)\,dz'\}/2r^2, \text{ \&c., \&c.}$$

158. Illumination through a Refracting System.

If S be a small area perpendicular to the axis of a coaxial refracting system and S' be its image, the same amount of light will fall on S' from S as on S from S', if their intrinsic brightness in air be the same.*

Let S and S' meet the axis of the system in Q and Q', and let the light from S, which passes through the image S', meet the unit planes through H and H' in an area Σ. If I be the intrinsic brightness of S in air, and μ be the index of refraction of the first medium, the total illumination of Σ by S, which is the amount of light that finally falls on S', is $\mu^2 IS\Sigma/QH^2$, since we must treat the cosines of the angles of divergence as unity.

Similarly if the intrinsic brightness of S' in air be I and μ' be the index of the last medium, the amount of light sent by S' to S is $\mu'^2 IS'\Sigma/Q'H'^2$.

But by (iv) Art. 77

$$\frac{S'}{S} = \left(\frac{H'Q'/\mu'}{HQ/\mu}\right)^2.$$

Hence the illuminations of one area by the other are equal.

In this theorem the loss of light due to absorption, reflection or refraction, has been neglected.

159. Apparent Brightness of Objects and Images.

When a luminous object is viewed, either directly or through an optical instrument, the light that enters the eye is spread over the image on the retina; and the intensity of illumination of the retina is the total light entering the eye divided by the area of the image. We assume that the stimulus to each element of the retina is proportional to the intensity of illumination; the estimate formed of the difference in brightness between two objects will then, by Fechner's law in physiology, be proportional to the logarithm of the ratio of the intensities they produce.

The eye cannot, however, distinguish differences below a certain smallness, nor can we estimate at all accurately the difference between two very bright lights.

* Helmholtz, *Physiol. Optik*, 1896, p. 209.

Let S be the area of the small object perpendicular to the line of sight, I its intrinsic brightness, and let S' be the area of the image on the retina, when the object is viewed directly. Then the light sent from S to S' is the same as that which would be sent from S' to S, if I were the intrinsic brightness of S'.

Let p be the radius of the pencil at the unit planes of the eye, which we may take as the radius of the pupil, and let v be the distance of the retina from the second unit plane, μ' the index of refraction of the vitreous humour; the light that would fall from S' on the unit planes and therefore ultimately on S is $\mu'^2 IS'\pi p^2/v^2$.

Hence the intensity of illumination of the retinal image is $\mu'^2 I\pi p^2/v^2$.

Neglecting any changes in v due to accommodation, this is independent of the position of the object; and therefore an extended luminous object appears equally bright at all distances.

The apparent brightness depends on the area of the pupil, and it is well known that vision in a dim light improves after a time; this is due to the dilatation of the pupil.

When the object is viewed through an optical instrument, that and the eye may be regarded as forming one system, and therefore the same theorem shews that the intensity of illumination of the retinal image is $\mu'^2 I\pi p'^2/v^2$, where p' is now the radius of the pencil as it enters the eye after passing through the instrument.

Since p' can never exceed p it follows that the image cannot appear brighter than the object seen directly; and if p' be less than p, the apparent brightness of the object is decreased.

Assuming that the eye is placed at the eye-ring of the instrument, p' is equal to the radius, ρ, of the eye-ring, and if y_0 be the radius of the object-glass, M the magnifying power of the instrument, $\rho = y_0/M$. Hence the intensity of illumination may be decreased in the ratio $y_0^2 : M^2p^2$; so that with high magnifying powers the general field of view and any extended objects, as comets or nebulae, are decreased in brightness when seen through a telescope. The larger the aperture of the telescope the less the decrease in brightness, and in seeking for comets a telescope of wide aperture but low magnifying power is used.

160. Apparent Brightness of Stars.

When an object is so small or so distant that its image is formed on one element only of the retina, we must consider the stimulus to the retina as proportional to the total amount of light that enters the eye. This is proportional to the solid angle subtended by the eye at the object; and is therefore directly proportional to the area of the pupil, and inversely proportional to the square of the distance of the object.

In the case of a star viewed with the naked eye, the light falls on the pupil of area πp^2; when the star is viewed through a telescope the light falls on the object-glass of area πy_0^2 and all the light finally enters the eye, provided that the radius of the pupil p be greater than the radius of the eye-ring ρ. If $p < \rho$, the fraction $\pi y_0^2 (p/\rho)^2$ of the light that falls on the telescope enters the eye. The stimulus to the retina is therefore increased in the lesser of the two ratios $y_0^2 : p^2$ or $y_0^2 : \rho^2$. An increase therefore in the aperture of telescopes will enable us to see much fainter stars; with the naked eye stars of the first six magnitudes are visible, with the largest telescopes at present in use the range is extended to the eighteenth magnitude.

Since in the first case, when $p > \rho$, the apparent brightness of the field is decreased in the ratio $\rho^2 : p^2$, and in the second case, when $p < \rho$, it is unaltered, it follows that in both cases the telescope increases the brightness of the star relative to that of the field in the ratio $y_0^2 : \rho^2$, or $M^2 : 1$. Hence with high magnifying powers it is possible to observe bright stars in the daytime.

In spite of what has been said above, it is doubtful how far intensity of illumination is a fitting measure of the subjective impression of apparent brightness, and the problem is really one of physiology. Thus it is stated that a large and small area having the same actual illumination apparently differ in brightness*.

Again, it is well known that it is possible to see a star in the daytime from the bottom of a high tower or chimney; the light from the star has not been increased, and the intensity of illumination of the retina from the sky remains the same. The only thing that has been decreased is the total light sent into the

* Abney, *Proc. Roy. Soc.* Vol. LXI. p. 331.

eye from the sky. The fact that in a dim light it is possible with spectacles or an opera-glass to distinguish objects and details of objects which are then quite indistinguishable to the naked eye, although they may be easily visible in a bright light, also seems to shew that the total amount of light that falls on the retina is an important factor in the subjective impression.

EXAMPLES.

1. A small plane area at a point P is illuminated by n bright points $A_1, A_2, \ldots A_n$ of intensities $\mu_1, \mu_2, \ldots \mu_n$. If the plane be placed so as to contain the normal at P to that member of the family of surfaces $\Sigma(\mu/r) = c$, which passes through P, the illumination at P is the same on both sides. But if it be placed so as to be tangential to the same surface, the difference of the illuminations on the two sides is a maximum; $r_1, r_2, \ldots r_n$ denoting the distances from $A_1, A_2, \ldots A_n$ respectively.

2. Two semi-infinite circular cylinders of equal intrinsic brightness but different radii stand with their bases on a horizontal plane; shew that the locus of points on the plane where they produce equal intensities of illumination is a circle.

3. A very long circular cylinder of radius a is uniformly bright. Shew that the illumination per unit area on a plane parallel to its axis at any point is $\pi I(a/c)\cos\phi$, where ϕ is the angle of incidence of the ray that leaves the cylinder normally, and c the distance of the point from the axis.

4. A ring is generated by any closed oval curve revolving about an axis in its own plane, which does not intersect it. Shew that, if the ring be luminous, and a small area be placed at any point on the axis perpendicularly to the axis, the illumination of that area is proportional to $\sin^2\phi_1 - \sin^2\phi_2$; where ϕ_1 and ϕ_2 are the extreme angles of incidence.

5. Two equal circular discs are placed horizontally with their centres in a vertical line at a distance apart equal to the radius of either and are illuminated by a uniformly bright sky. Prove that the illumination at the centre of the lower disc is one half the illumination at any point of the upper disc.

6. A luminous sphere of radius a is placed inside a hemisphere of radius b, the centre of the sphere being on the axis of the hemisphere at distance c from its centre. Shew that, if the base of the hemisphere be perfectly reflecting, the illumination at a point of its curved surface at an angular distance θ (where $\sin\theta > a/c$) from the axis varies as

$$(b - c\cos\theta)(b^2 - 2bc\cos\theta + c^2)^{-\frac{3}{2}} + (b + c\cos\theta)(b^2 + 2bc\cos\theta + c^2)^{-\frac{3}{2}}.$$

7. Two uniformly bright vertical walls of the same uniform height and of infinite length intersect at right angles; shew that the intensity of illumination at a point on the ground is

$$\tfrac{1}{2}I[\tfrac{3}{2}\pi - \cos\alpha\,\{\pi - \tan^{-1}(\tan\beta\operatorname{cosec}\alpha)\} - \cos\beta\,\{\pi - \tan^{-1}(\tan\alpha\operatorname{cosec}\beta)\}]$$

where I is the intrinsic brightness of the walls, and α and β are the greatest angles subtended at the point by vertical lines drawn on the walls.

8. The angular radius of the sun is α, and its centre is at distance β below the horizon. Shew that the illumination per unit area on a vertical wall facing the sun is proportional to $(\gamma - \sin\gamma\cos\gamma)\sin^2\alpha\cos\beta$, where $\cos\gamma = \tan\beta\cot\alpha$. Shew that the illumination on the ground is proportional to

$$\cos^{-1}(\cos\alpha\sec\beta) - \gamma\sin^2\alpha\sin\beta - \sin\alpha\cos\alpha\cos\beta\sin\gamma.$$

9. A uniformly bright cone of revolution of semi-vertical angle α stands on a plane. Shew that at any point of the plane, where the altitude of the vertex of the cone is θ, the illumination per unit area of the plane is proportional to

$$\sin^{-1}(\tan\alpha\tan\theta) - \sin\alpha\sin^{-1}(\sin\theta\sec\alpha);$$

and the illumination per unit area of a vertical plane making an angle ϕ (where $\sin\phi > \tan\alpha\tan\theta$) with the vertical plane through the axis is proportional to $\sin\alpha\tan\theta\sin\phi\sin^{-1}(\sin\theta\sec\alpha)$.

10. An infinitely long bright circular cone of semi-vertical angle α lies on a plane. Shew that the intensity of illumination at any point of the plane is proportional to

$$(\pi - \theta)\sin^2\psi + \chi\sin^3\alpha + \sin\alpha\cos^2\alpha\sin\chi\cos\chi,$$

where θ is the angle between the line of contact of the cone and the radius vector from the vertex to the point, and ψ and χ are given by the equations $\cot\psi = \cot\alpha\sin\theta$, $\cot\chi = \cot\theta\sin\alpha$.

11. The surface of a given ellipsoid is uniformly self-luminous of intrinsic brightness I. Prove that the intensity of illumination of the interior surface of any greater confocal ellipsoid at any point is $\pi I\{\lambda_2\lambda_3/(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\}^{\frac{1}{2}}$ where λ_1, λ_2, λ_3 are the parameters of the three confocals at that point with regard to the bright ellipsoid.

12. A bright sphere rolls rapidly on a table round a circle. Shew that the curves of equal apparent illumination are concentric circles, the apparent illumination at any one being

$$\frac{2}{\pi}I'\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

where $(1-k^2)^{\frac{3}{2}} = I'/I$, and I and I' would be the illuminations at the points on the circle nearest to and furthest from the sphere if it were at rest.

13. A small area at a point P is illuminated by a circular disc, centre O, and radius a. Shew that, if the area be perpendicular to OP, the intensity of illumination is $\pi Ia^2\cos\theta/r_1r_2$, where θ is the angle between OP and the normal to the disc, r_1, r_2 the greatest and least distances of P from its boundary.

Shew that, if the small area be parallel to the disc, the intensity of illumination is $\pi I\{4a^2-(r_1-r_2)^2\}/4r_1r_2$.

14. A bright circular area is placed vertically, its centre C being at height h above a horizontal plane. Shew that, if P be any point in this plane, PM perpendicular to the plane of the area, the intensity of illumination of the horizontal plane at P is $\frac{1}{4}\pi Ih\dfrac{PM}{CM^2}\dfrac{(r_1-r_2)^2}{r_1r_2}$, where r_1, r_2 are the greatest and least distances of P from the boundary of the circle.

15. Prove that the mutual illumination of two surfaces is equal to $\frac{1}{2}I\iint\cos\theta\cos\theta'\,ds\,ds'$, taken round their boundaries, where θ, θ' are the angles made by the elements ds, ds' with the line joining them.

Two circular areas are placed on the same axis, their planes being perpendicular to the axis. Shew that their mutual illumination is $\frac{1}{4}\pi^2 I(r_1-r_2)^2$, where r_1, r_2 are the greatest and least distances between any two points, one on each boundary.

16. Two equal rectangular areas of sides a, b are placed so that corresponding sides are parallel and the distance c between corresponding corners is perpendicular to their planes. Shew that their mutual illumination is

$$I\left[c^2\log\left\{\frac{(c^2+a^2)(c^2+b^2)}{(c^2+a^2+b^2)\,c^2}\right\}+2a(c^2+b^2)^{\frac{1}{2}}\tan^{-1}\frac{a}{(c^2+b^2)^{\frac{1}{2}}}-2ac\tan^{-1}\frac{a}{c}\right.$$
$$\left.+2b(c^2+a^2)^{\frac{1}{2}}\tan^{-1}\frac{b}{(c^2+a^2)^{\frac{1}{2}}}-2bc\tan^{-1}\frac{b}{c}\right].$$

17. Prove that, if S and S' be any small areas perpendicular to the axis of a coaxial refracting system, and separated by the system, as much light will pass from S to S' as from S' to S, if their intrinsic brightnesses in air be equal.

18. Rays from a source of light on the axis of a symmetrical optical instrument and in front of it are received on a screen behind it. Compare the illumination due to the transmitted light at the centre of the screen with that which would be due to directly incident light if the instrument were not there. Shew that for suitable positions of the source there may be two positions of the screen for which the illumination is unaltered, loss of light in refraction being neglected.

19. Determine the effect on the brightness of an image formed by a telescope of magnifying power m, which is produced by placing a circular patch of radius r at the centre of the object-glass in the cases when the radius of the pupil of the eye is (i) $>r/m$, (ii) $<r/m$.

20. Prove that in the photography of stars the exposure required varies inversely as the square of the linear aperture of the object-glass: while in the case of nebulae and other continuous objects it varies inversely as the square of the angular aperture.

CHAPTER X.

GENERAL THEOREMS.

161. Fermat's Theorem.

The principles on which the theory of Geometrical Optics is based are the laws of reflection and refraction; these are summed up in a theorem due to Fermat, known as the Principle of Least Time, from which can be deduced all the properties of pencils of rays caused by reflection or refraction.

Let a ray of light pass from one given point to another through any number of media, being reflected or refracted at any surfaces, and let r be the length of the part of the path in a medium of refractive index μ; the expression $\Sigma\mu r$ is called the *reduced path* from the one point to the other.

[If the ray only pass virtually through the final point, the distance r in the last medium to that point must be taken as a negative quantity.]

The reduced path is stationary in value for small displacements of the points of incidence of the ray on the reflecting or refracting surfaces.

Let ds be a small arbitrary displacement of the point of incidence on a surface separating media of indices μ and μ'; let the parts of the ray in those media make angles θ, θ' with the direction of ds; we have by Art. 18

$$\mu \cos \theta = \mu' \cos \theta'.$$

But if r be the distance from a fixed point on the incident ray to the point of incidence, and r' the distance from the point of incidence to a fixed point on the refracted ray,

$$\cos \theta = \frac{dr}{ds}; \text{ and } \cos \theta' = -\frac{dr'}{ds}.$$

Hence the reduced path is stationary in value for such a displacement.

If the ray be reflected, the distances r and r' are measured in the same medium, and $\theta = \theta'$ (Art. 3); hence the reduced path is again stationary in value.

When the ray is refracted at successive surfaces, the displacements of the points of incidence on the several surfaces may be considered independently; and hence, the initial and final points on the ray being fixed, the reduced path is stationary, and the equations that express this condition will completely determine the path of the ray.

In the case of a medium whose index of refraction varies continuously from point to point, we can determine the path of a ray by making the integral $\int \mu ds$, taken between definite limits, stationary in value.

Since on the undulatory theory of light the refractive index of any medium is inversely proportional to the velocity of light in that medium, the reduced path between two points is proportional to the time taken by the light in travelling from one to the other.

The time therefore along the path of the ray differs from that along any arbitrary adjacent path by small quantities of the second order; and is a true minimum, as we shall see below (Art. 198), in all cases where the final point on the ray is taken before the two points of contact of that ray with the caustic surface, which is the envelope of rays diverging from the initial point.

162. Theory of Geometrical Foci.

Let a spherical surface separate two media of indices μ and μ', and let the equation of the surface referred to its vertex as origin be $\xi^2 + \eta^2 + \zeta^2 = 2\rho\zeta$, the axis of z being drawn into the first medium.

Let (x, y, z) be the coordinates of a point P, which we suppose near the axis of z, let (x', y', z') be those of P', also near the axis; and let a ray from P be refracted at the point R and pass through P'.

Then in Fermat's theorem, $r = PR$, $r' = RP'$, while

$$\begin{aligned} r &= \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{\frac{1}{2}} \\ &= \{z^2 + x^2 + y^2 - 2(x\xi + y\eta) - 2(z-\rho)\zeta\}^{\frac{1}{2}} \\ &= z\left\{1 + \frac{x^2+y^2}{2z^2} - \frac{x\xi + y\eta}{z^2} - \frac{z-\rho}{z^2}\,\frac{\xi^2+\eta^2}{2\rho}\right\}, \end{aligned}$$

if we substitute for ζ from the equation of the surface, and retain only the squares and products of the small terms x, y, ξ and η.

Since r' is measured onwards from the refracting surface into the second medium, *i.e.* in the opposite direction to the axis of z, we must put in this case

$$r' = -z'\left\{1 + \frac{x'^2 + y'^2}{2z'^2} - \frac{x'\xi + y'\eta}{z'^2} - \frac{z' - \rho}{z'^2}\,\frac{\xi^2 + \eta^2}{2\rho}\right\}.$$

Hence, making $\mu r + \mu' r'$ stationary in value for variations of ξ and η, we have

$$\mu\frac{x}{z} - \mu'\frac{x'}{z'} = \left(\mu\frac{z-\rho}{z} - \mu'\frac{z'-\rho}{z'}\right)\frac{\xi}{\rho},$$

$$\mu\frac{y}{z} - \mu'\frac{y'}{z'} = \left(\mu\frac{z-\rho}{z} - \mu'\frac{z'-\rho}{z'}\right)\frac{\eta}{\rho}.$$

When the points P and P' are taken arbitrarily, these equations give uniquely the point on the surface at which a ray will be refracted from P to P'; but if their coordinates are connected by the equations

$$\mu\left(\frac{1}{\rho} - \frac{1}{z}\right) = \mu'\left(\frac{1}{\rho} - \frac{1}{z'}\right),$$

$$\mu\frac{x}{z} = \mu'\frac{x'}{z'}, \quad \mu\frac{y}{z} = \mu'\frac{y'}{z'},$$

every ray from P within the limits of approximation adopted will be refracted to pass through P'. These are the formulae of Chapter III. for the position of the geometrical focus, and for the linear magnification; and all the theorems hitherto obtained for small pencils passing directly through a coaxial refracting system might be deduced from them.

163. The Abbe-Helmholtz law of sines for pencils of wide aperture.

The relation between the sines of the angles of divergence of wide-angled pencils, of which a particular case was given in Art. 47, has been extended, by means of Fermat's theorem, to any aplanatic system of refracting surfaces.

Let the system of refracting surfaces bring the rays from a point Q in the first medium of index μ accurately to a focus Q' in

the last medium of index μ'. Let the system also bring the rays from an adjacent point P accurately to a focus at P', *where* PQ *and* $P'Q'$ *are perpendicular to the path of a ray* $QA\,..\,A'Q'$, which we call the axial ray.

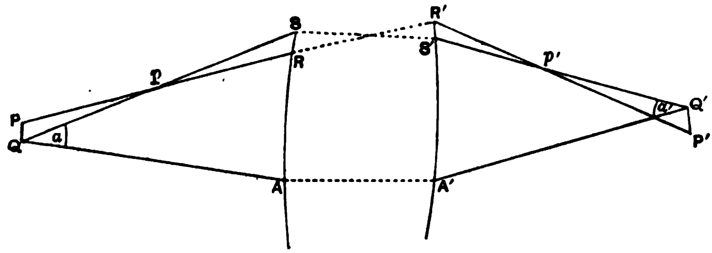


Fig. 81.

Let α, α' be the finite angles of divergence of a ray $QS\,..\,S'Q'$ from the axial ray, and let l, l' be the linear dimensions of PQ, $P'Q'$. These quantities are connected by the equation

$$\mu l \sin\alpha = \mu' l' \sin\alpha'.$$

Let $PR\,..\,R'P'$ be the path of an adjacent ray, meeting the ray $QS\,..\,S'Q'$ in p and p'.

Since the reduced path from Q to Q' is the same for all rays, and similarly the reduced path from P to P' is the same for all rays, the difference between these for the rays $QS\,..\,S'Q'$ and $PR\,..\,R'P'$ is a constant. But the reduced path from p to p' is the same for the adjacent rays $pR\,..\,R'p'$ and $pS\,..\,S'p'$.

Hence $\quad \mu(Qp - Pp) + \mu'(p'Q' - p'P')$ is constant,

i.e. $\quad \mu l \sin\alpha - \mu' l' \sin\alpha'$ is constant.

But, by hypothesis, when α is zero α' is also zero; hence the constant is zero, and the relation between the sines of the angles of divergence is as stated above. The proof shews that the reduced paths from Q to Q' and from P to P' are equal.

[The relation above cannot hold for $\alpha = \frac{1}{2}\pi$ unless $\alpha' = \frac{1}{2}\pi$ at the same time; for if QP be the direction of a possible ray, that ray must emerge as $Q'P'$; hence $\mu l = \mu' l'$ and $\alpha' = \alpha$ for all values, as in a plane mirror.]

This condition of aplanatism is practically reached in good microscopes, when the object is perpendicular to the axis at a

certain point. This point is extremely close to the first surface of the objective and the angle α may be very nearly a right angle; the angle α' however at emergence will not exceed a few degrees, since the rays have to enter the pupil of the eye.

In the microscope the magnifying power is equal to l'/l or $\mu \sin \alpha / \sin \alpha'$; thus with the same angles of divergence α and α', the magnifying power is increased by the method of homogeneous immersion, where the object is placed in a medium of refractive index μ greater than unity.

164. Curves of Illumination.

If light from a single origin fall on a series of fine polished wires or grooves, an observer will see a series of bright points on the wires, passing into a continuous curve as the distances between the wires are decreased. Such curves are called *curves of illumination.*

This fact may be explained as follows. At any point of a wire there will be an infinite number of tangent planes all passing through the tangent line, and these will reflect an incident ray along the generators of a cone having the tangent line as axis. If the point be so chosen that the observer's eye is on a generator of this cone, he will see a bright image of the origin of light.

We may therefore determine the required point P on the wire by finding the condition that the lines OP, PE make equal angles with the tangent line, where O is the origin of light, and E the eye. It will then always be possible to determine one of the infinite number of normals to the wire at P, which shall lie in the plane OPE and also bisect the angle between OP and EP.

This point may be more easily determined analytically by the use of Fermat's theorem; the curve of illumination may then be found. For the coordinates of P may be expressed in terms of two parameters, (i) β_1, determining the wire on which it lies, (ii) β_2, determining its position on the wire.

If $OP + PE$ be made stationary in value for variation of β_2, the point P on the wire defined by β_1 which reflects light to the eye has been found; and the elimination of β_1 between the coordinates of P will give the curve of illumination.

165. Example.

A bicycle wheel in which the spokes are perpendicular to the axis is placed in the sun and spun rapidly. Shew that the equation of the bright curve seen on the spokes by an eye in the axis of the wheel produced is of the form

$$r^2(\sec^2\theta\sec^2\alpha-1)=a^2,$$

α denoting the angle between the direction of the sun's rays and the plane of the wheel, and a the distance of the eye from the wheel.

In a case like this, where the origin of light is practically at infinity, it is plain that the variable part of the distance OP is the normal distance from any fixed plane perpendicular to the sun's rays to the point P, taken positively in the direction of the rays.

If we take the centre of the wheel as origin, the plane through the eye and the sun as the plane of zx, and the axis of the wheel as the axis of z, the equation of a plane perpendicular to the sun's rays is $x\cos\alpha+z\sin\alpha=0$, and since the coordinates of any point P on a spoke may be written as $r\cos\theta$, $r\sin\theta$, 0, the variable part of $OP+PE$ is equal to

$$-r\cos\theta\cos\alpha+(a^2+r^2)^{\frac{1}{2}}.$$

If this be made a minimum for variation of r, θ being constant, we have

$$r/(a^2+r^2)^{\frac{1}{2}}=\cos\theta\cos\alpha,$$

i.e. the bright curve seen on the spokes is the half of the curve

$$r^2(\sec^2\theta\sec^2\alpha-1)=a^2,$$

which lies on the positive side of the axis of y.

EXAMPLES.

1. There are two ports P and Q at distances a and b from a promontory, the two coasts being supposed straight and meeting at an angle α. Find the quickest route first by sea and then by land from P to Q, and shew that if any of the route is by sea the distance traversed is

$$a+b\left(\sqrt{\frac{u-v}{u+v}}\sin\alpha-\cos\alpha\right),$$

where u and v are the velocities by sea and by land.

Shew that, if b lie between $a(u^2-v^2)^{\frac{1}{2}}/\{\cos\alpha(u^2-v^2)^{\frac{1}{2}}\pm v\sin\alpha\}$, the route is wholly by land.

2. If a series of fine smooth grooves be cut in a plane surface in the shape of concentric circles, the bright curve formed by reflection of the light from a luminous point and seen by an eye in the plane through the luminous point and the axis of the circles will be a circle.

3. A series of parallel straight grooves are drawn on a plane. Shew that the curve of illumination due to the presence of a bright point is part of a cubic curve; and trace the visible curve of illumination in the case, where the eye and the bright point lie in a plane through one of the grooves perpendicular to the plane of the grooves.

4. A reflecting plane is striated by confocal and coaxial parabolas. Shew that the bright curve seen in sunlight by an eye on the normal to the plane through the focus is a circle, the eye and the sun lying in a plane through the axis of the parabolas.

5. The surface of a hollow right circular cone is grooved with an infinite number of circular grooves. A bright point is placed on the surface; prove that an eye on the opposite generating line will see a bright curve, which lies on a sphere of radius $abc/(a^2 \sim b^2)$ passing through the vertex of the cone, where a and b are the distances from the vertex of the bright point and the eye, and c is the distance between them.

6. Fine polished wire with circular transverse section is disposed along the meridians of a sphere whose axis is directed to the sun. Prove that those reflected rays which have a common direction normal to the axis proceed from the curves in which the sphere is met by an elliptic cone, the planes of whose circular sections are respectively perpendicular to the incident light, and to the bisector of the angle between the incident and reflected light.

7. In a hollow ellipsoidal shell small polished grooves are made coinciding with one series of circular sections, and a bright point is placed at one of the umbilics in which the series terminates; prove that the locus of the bright points seen by an eye in the opposite umbilic is a central section of the ellipsoid, and that the whole length of the path of any ray between the two umbilics is the same.

8. If a polished wire in the shape of an epicycloid of small circular cross-section, generated by the rolling of a circle of radius b on another of radius a, be rotated rapidly about an axis through its centre, perpendicular to its plane, and if the eye, *supposed distant*, and the sun be in a plane containing the axis of rotation, the appearance presented will be that of a bright ellipse, whose semi-axes are $a+2b$ and a.

9. A man sees the light of a star reflected on the surface of the sea, which is covered with small ripples travelling in all directions. If these be such that at no point is the normal to the surface inclined to the vertical at an angle greater than β, shew that the boundary of the bright patch on the water is the curve in which the surface is cut by the cone

$$\cos\alpha+\cos\theta=2\cos\beta\sin\tfrac{1}{2}\delta,$$

where α is the zenith distance of the star, θ the angle between a reflected ray and the vertical, and δ the deviation of the ray at reflection.

10. A system of refracting surfaces is aplanatic for two pairs of points P and P', Q and Q', all lying on the same axial ray, and PQ and $P'Q'$ are small lengths l and l'. Shew that if α, α' be the angles of divergence of any ray from the axial ray

$$\mu l\sin^2\tfrac{1}{2}\alpha=\mu' l'\sin^2\tfrac{1}{2}\alpha'.$$

A thin lens of focal length f in which the curvatures of the faces are in the ratio $x+1 : x-1$ is approximately aplanatic for a small object lying along the axis at distance $f\Big/\left\{\dfrac{(\mu+1)\mu}{(\mu-1)x}-\dfrac{1}{2}\right\}$ from the lens, if $x^2=\mu(3\mu+2)$.

11. Shew that if A, B, C be the angular points of an equilateral triangle in a medium of index μ, and if after refraction through any number of media a, b, c be perfect images of A, B, C, and form another equilateral triangle in a medium of index μ', the ratio of AB to ab will be

$$\mu' : \mu \text{ or } 3\mu' : \mu \text{ or } \mu' : 3\mu.$$

MALUS' THEOREM.

166. *Any system of rays, which are originally normals to a surface, will retain the property of being normals to a surface after any number of reflections or refractions.*

Let $ABCD$, $A'B'C'D'$ be the paths of two near rays, which are normals to a surface at A and A', and which are afterwards refracted or reflected, as the case may be, at B and C, B' and C'.

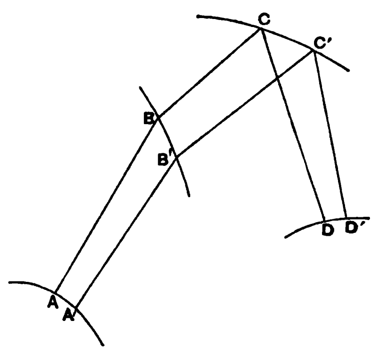


Fig. 82.

Take two points D, D' on their final directions so that the *reduced path* $\Sigma\mu r$ may be the same for $ABCD$ and $A'B'C'D'$.

Then since $\Sigma\mu r$ taken along the actual path $ABCD$ of a ray is stationary in value for small displacements of B and C on the refracting surfaces, it has the same value if taken along $AB'C'D$.

Hence $\Sigma\mu r$ is the same for $AB'C'D$ and for $A'B'C'D'$. But since AA' is orthogonal to the rays, AB' is equal to $A'B'$. Hence $C'D$ must be equal to $C'D'$, and therefore DD' is orthogonal to the final directions of the rays. If A' be taken in all positions near A on the orthotomic surface, D' will describe an element including D of a surface orthotomic to the rays. It is therefore possible, when the shape of the original orthotomic surface is given, to obtain the orthotomic surface of the rays after any refractions or reflections by calculating the reduced path along any ray.

The wide generality of this theorem would justify our taking it as the fundamental theorem of Geometrical Optics; and as we see in the following articles, its applications are very varied.

167. Characteristic Function.

Given an initial orthotomic surface of the rays, the value of the reduced path $\Sigma\mu r$, taken from that surface to any point P, is known as the *Characteristic Function* V, where V is supposed to be *expressed in terms of the constants of that surface and the coordinates of* P *alone*, the coordinates of the various points of refraction having been eliminated by application of Fermat's theorem.

The reduced path between any two points on a ray, or between the orthotomic surfaces through those points, will be the difference of the values of V for those points. The properties of a pencil of rays, originally normals to a surface, are completely determined by the existence of this function. Since however we confine ourselves in this chapter to small pencils of light incident on given refracting surfaces, it seems simpler to base the equations obtained on Sturm's properties of normals and on Fermat's theorem, and to defer the consideration of the general properties of the Characteristic Function to the next chapter.

On the principles of the Undulatory Theory, in which the fundamental idea is the existence of a wave-front, *i.e.* a surface such that the disturbance of the ether at all points on it is in the same phase simultaneously, and rays are defined as the lines of propagation of the disturbance, Malus' theorem is seen intuitively.

If an original wave-front be given, all consecutive wave-fronts may be determined by the fact that the time of propagation of the disturbance from one front to another is the same for all points on those surfaces; in other words, the reduced path from one orthotomic surface to the other is a constant.

It may be mentioned that lines, forming a congruence and defined by the property that through any point there is only one line, are not necessarily normals to a surface; moreover the rays are normals to the wave-front only in media in which the velocity of propagation at any point is the same in all directions; in doubly-refracting media, as crystals, the velocity of propagation depends on the direction of the ray relatively to the axes of the crystal, and the extraordinary rays are not normals to a surface.

168. Example.

To find the orthotomic surfaces after refraction at a plane of a pencil of light diverging from a point.

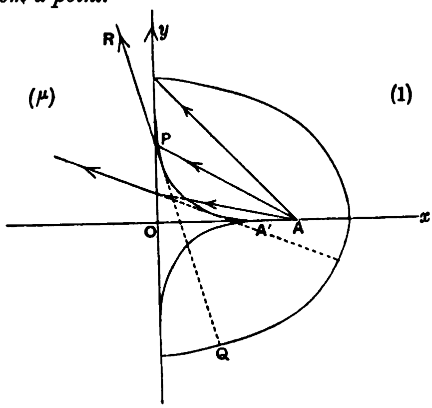


Fig. 83.

Let A be the origin of light, AP any ray incident at angle ϕ on the plane bounding the medium of relative refractive index μ.

If we take $AP+\mu PR=c$, where c is any arbitrary constant, all the refracted rays are orthogonal to the locus of R.

If $OA=a$, this equation gives for the coordinates of R

$$x=-PR\cos\phi'=-\frac{1}{\mu}(c-a\sec\phi)\cos\phi',$$

$$y=a\tan\phi+PR\sin\phi'=a\tan\phi+\frac{1}{\mu}(c-a\sec\phi)\sin\phi'.$$

The elimination of ϕ and ϕ' between these equations, with the equation $\sin\phi=\mu\sin\phi'$, gives the locus of R.

As a particular case, put c zero, *i.e.* let a point Q be taken on the refracted ray at distance AP/μ behind the point of incidence.

The coordinates of Q are

$$x=\frac{a\cos\phi'}{\mu\cos\phi},\qquad y=a\tan\phi-\frac{a\sin\phi'}{\mu\cos\phi}=a\left(\mu-\frac{1}{\mu}\right)\frac{\sin\phi'}{\cos\phi}.$$

Hence $$x^2-\frac{y^2}{\mu^2-1}=\frac{a^2}{\mu^2\cos^2\phi}\{\cos^2\phi'-(\mu^2-1)\sin^2\phi'\}=\frac{a^2}{\mu^2}.$$

One orthotomic surface is therefore the prolate quadric generated by the revolution of this conic about the axis of x; the others are surfaces parallel to this; and all the refracted rays touch a caustic surface, generated by the revolution of the evolute of this conic about the axis of x, the cusp of the evolute being the geometrical focus of A.

The eccentricity of the conic is μ, so that it is an hyperbola or ellipse as the second medium is more or less refractive than the first; and the origin of light is one of its foci.

If as in the figure the rays pass from a denser to a rarer medium, the caustic is the evolute of an ellipse; the ray incident at the end of the minor axis is incident at the critical angle, and emerges along the surface of separation.

169. Dispersion.

We may apply Malus' theorem to determine the dispersion produced by any system of refractive media in a ray of white light, supposed incident and emergent in air.

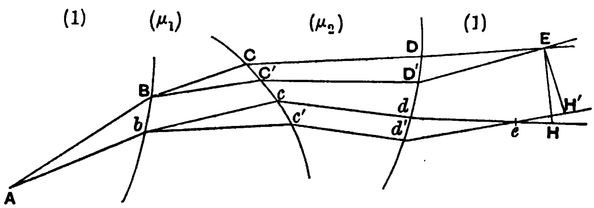


Fig. 84.

Let a ray of index μ pursue the path $ABCDE$, and the ray of slightly different index $\mu+\partial\mu$ the path $ABC'D'E$, these two coloured rays meeting at E after emergence. For convenience the path is taken through two media, but the proof is independent of the number of media. Let an adjacent ray of index μ pursue a path $Abcde$, and the other ray of index $\mu+\partial\mu$ the path $Abc'd'e$. Draw EH perpendicular to the ray DE, and EH' perpendicular to the ray $D'E$.

Then by Malus' theorem

$$AB+\mu_1 BC+\mu_2 CD+DE=Ab+\mu_1 bc+\mu_2 cd+dH \qquad \text{(i)},$$

$$AB+(\mu_1+\partial\mu_1)\,BC'+(\mu_2+\partial\mu_2)\,C'D'+D'E$$
$$=Ab+(\mu_1+\partial\mu_1)\,bc'+(\mu_2+\partial\mu_2)\,c'd'+d'H' \qquad \text{(ii)}.$$

Hence on subtraction

$$\partial\mu_1 BC'+\partial\mu_2 C'D'+\mu_1(BC'-BC)+\mu_2(C'D'-CD)+D'E-DE$$
$$=\partial\mu_1 bc'+\partial\mu_2 c'd'+\mu_1(bc'-bc)+\mu_2(c'd'-cd)+d'H'-dH \qquad \text{(iii)}.$$

But by Fermat's theorem, since a ray of index μ passes from B to E,

$$\left.\begin{aligned}\mu_1(BC'-BC)+\mu_2(C'D'-CD)+D'E-DE&=0\\ \text{and}\qquad \mu_1(bc'-bc)+\mu_2(c'd'-cd)+d'e-de&=0\end{aligned}\right\} \qquad \text{(iv)}.$$

Also $\quad d'H'-dH=d'e-de+eH'-eH=d'e-de+EH\,(H\hat{E}H'),$

where the angle HEH' is the chromatic dispersion $\partial\psi$ between the emergent rays DE and $D'E$.

Hence in the limit we may write (iii) in the form

$$\partial\psi=\{(\partial\mu_1 BC+\partial\mu_2 CD)-(\partial\mu_1 bc+\partial\mu_2 cd)\}/EH \qquad \text{(v)}.$$

As the position of the origin A is immaterial to equation (v), we may suppose it chosen so that the emergent rays DE and dH are parallel; and therefore in any system of refractive media bounded by air on both sides, the chromatic dispersion of a ray is equal to the ratio of the difference of the values of the function $\Sigma\partial\mu r$, taken through the media for the two sides of a thin emergent beam, to the breadth of the emergent beam.

170. Primary and Secondary Foci.

In the following articles we confine our attention to small pencils, in which the paths of the rays deviate but slightly from the path of some one ray, which is known as the axial ray.

The areas in which the rays meet their orthotomic surfaces or the refracting surfaces are therefore small throughout, and we may, as in Euler's theorem, regard these surfaces in the neighbourhood of the axial ray as paraboloids. It will be necessary first to put in evidence the properties of adjacent normals to a surface, known as Sturm's theorems.

Let the axis of z be that of the pencil, and let the equation of the orthotomic surface at the origin be

$$z = \tfrac{1}{2}(x^2/v_1 + y^2/v_2) + u_3 + \ldots.$$

The equations of the normal to the surface at the point (x, y, z) are

$$(\xi - x)\Big/\left(\frac{x}{v_1} + \frac{\partial u_3}{\partial x}\right) = (\eta - y)\Big/\left(\frac{y}{v_2} + \frac{\partial u_3}{\partial y}\right) = (\zeta - z)/(-1).$$

Putting $\xi = 0$, we see that this line cuts the plane of yz in a point lying, to a first approximation, on the straight line $\zeta = v_1$. Similarly the ray meets the plane xz in a point lying on the straight line $\zeta = v_2$.

The two points on the axis at distances v_1, v_2 from the origin are the centres of curvature of the principal normal sections of the orthotomic surface; in optics they are known as the *primary and secondary foci*; the planes of reference, which are the principal planes of curvature, are known as the *primary and secondary focal planes*; and the small straight lines through the foci, as the *primary and secondary focal lines.* Each focal line is at right angles to the corresponding focal plane.

The approximations of the theory of primary and secondary foci are not so close as those of the theory of geometrical foci; the longitudinal aberration of an adjacent ray from the primary or secondary focus is proportional to the first power of the aperture, the aberration of an adjacent ray from the geometrical focus was found to be proportional to the square of the aperture; the lateral aberrations are respectively of the second and third orders.

171. Normal distance.

To obtain an expression for the normal distance from a given point to a surface, whose equation is given in the neighbourhood of the foot of the normal.

We have just seen that, to the first order of approximation, all the rays intersect two focal lines, which are therefore common to all the orthotomic surfaces of the rays in the same medium.

First, let the equation of the orthotomic surface at the origin be approximately

$$z = \tfrac{1}{2}(x^2/v_1 + y^2/v_2),$$

referred to the axes of the pencil.

Then the equation of the orthotomic surface at constant normal distance r is

$$z - r = \tfrac{1}{2}\{x^2/(v_1 - r) + y^2/(v_2 - r)\},$$

since this surface cuts the axis of z at distance r from the origin, and has the required focal lines.

Hence, if (ξ, η, ζ) be the coordinates of any point on this surface lying near the axis, so that ξ and η are small quantities,

$$r = \zeta - \tfrac{1}{2}\left(\frac{\xi^2}{v_1 - \zeta} + \frac{\eta^2}{v_2 - \zeta}\right) \quad\text{.................. (i)},$$

to our order of approximation.

If we take the light as moving in the negative direction along the axis of z (so that curvature concave to light is positive), then r is the normal distance with the light from the point to the orthotomic surface. The value of r is apparently infinite if $\zeta = v_1$ or v_2, *i.e.* if the point lie in a plane perpendicular to the axis at either of the foci; but at each focus all the normals meet only the corresponding focal line, and therefore if $\zeta = v_1$, we must have $\xi = 0$, and if $\zeta = v_2$, $\eta = 0$.

Secondly, let the axes of reference be turned about the axis of z so that the equation of the orthotomic surface at the origin takes the form

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2),$$

the axis of z being still the axis of the pencil.

Then $\qquad 1/v_1 + 1/v_2 = a + b, \quad 1/v_1v_2 = ab - h^2$

Hence the expression above for r, which may be written

$$r = \zeta - \tfrac{1}{2}\left\{\frac{\xi^2/v_1 + \eta^2/v_2 - (\xi^2 + \eta^2)\,\zeta/v_1v_2}{1 - \zeta\,(1/v_1 + 1/v_2) + \zeta^2/v_1v_2}\right\},$$

when referred to the axes of the pencil, will by the same change of axes be transformed into

$$r = \zeta - \tfrac{1}{2}\,\frac{a\xi^2 + 2h\xi\eta + b\eta^2 - (ab - h^2)(\xi^2 + \eta^2)\,\zeta}{1 - (a + b)\,\zeta + (ab - h^2)\,\zeta^2} \quad \text{... (ii).}$$

For *points in the immediate neighbourhood of the origin such that ζ is a small quantity of at least the same order as ξ and η*, it is obvious that the normal distance is given, to the order of approximation adopted in the form of the orthotomic surface, by the equation

$$r = \zeta - \tfrac{1}{2}\,(a\xi^2 + 2h\xi\eta + b\eta^2) \quad \text{............... (iii).}$$

Thirdly, when a pencil is incident obliquely on a refracting surface, it is necessary to express r in terms of coordinates referred to other axes than those of the pencil.

Let the axis of a small pencil be incident at the origin, the angle of incidence being ϕ; let the axis of the pencil be the axis of z, and the normal to the surface the axis of ζ. We take the plane of incidence as the plane of zx and of $\zeta\xi$, so that the axes of y and η coincide.

Then since the coordinates of any point referred to these axes are connected by the equations

$$x = \xi\cos\phi - \zeta\sin\phi,$$
$$z = \xi\sin\phi + \zeta\cos\phi,$$
$$y = \eta,$$

the equation of the orthotomic surface of the rays at the origin,

$$z = \tfrac{1}{2}\,(ax^2 + 2hxy + by^2),$$

transforms into

$$\xi\sin\phi + \zeta\cos\phi = \tfrac{1}{2}\,\{a\,(\xi\cos\phi - \zeta\sin\phi)^2 + 2h\,(\xi\cos\phi - \zeta\sin\phi)\,\eta + b\eta^2\},$$

and therefore the normal distance r from any point of current coordinates (ξ, η, ζ), *which is near the origin and such that ζ is of the second order in ξ or η*, is given by the equation

$$r = \xi\sin\phi + \zeta\cos\phi - \tfrac{1}{2}\,(a\xi^2\cos^2\phi + 2h\xi\eta\cos\phi + b\eta^2) \quad \text{...... (iv).}$$

172. *A small pencil is refracted at a given surface; to determine the constants of the refracted pencil, given those of the incident pencil.*

Let the angles of incidence and refraction of the axis of the pencil be ϕ and ϕ'; let the origin be the point of incidence, the axes Oz, Oz' the axes of the incident and refracted pencils respectively, and let the plane of incidence and refraction of the axis be the plane of zx for the incident pencil, the plane of $z'x'$ for the refracted pencil, and the plane of $\zeta\xi$ for the refracting surface, as in the previous article.

Let the equation of the orthotomic surface at the origin of the incident light be

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2),$$

the equation of the orthotomic surface of the refracted light

$$z' = \tfrac{1}{2}(a'x'^2 + 2h'x'y + b'y^2),$$

and the equation of the refracting surface

$$\zeta = \tfrac{1}{2}(R\xi^2 + 2S\xi\eta + T\eta^2).$$

Since, by Malus' theorem, the reduced path between the two orthotomic surfaces is the same whatever ray be chosen, we have

$$\mu r + \mu' r' = \text{its value for the origin} = 0.$$

The normal distances r, r' from the orthotomic surfaces of the point (ξ, η, ζ) of incidence of a ray must be taken with their proper signs; and since in Malus' theorem the first distance r is measured from the first orthotomic surface to the point of incidence, and the second distance r' from the point of incidence to the second orthotomic surface, in both cases onwards with the light, we must give to the expression for r the opposite sign to that obtained in (iv) Art. 171, and to r' the same sign as in (iv). Hence substituting from this equation, we have to this order of approximation,

$$\mu\,\{\xi \sin\phi + \zeta\cos\phi - \tfrac{1}{2}(a\xi^2\cos^2\phi + 2h\xi\eta\cos\phi + b\eta^2)\}$$
$$= \mu'\,\{\xi\sin\phi' + \zeta\cos\phi' - \tfrac{1}{2}(a'\xi^2\cos^2\phi' + 2h'\xi\eta\cos\phi' + b'\eta^2)\}$$

for all points on the refracting surface.

On substituting for ζ from the equation of the surface, and equating the coefficients of ξ, ξ^2, η^2, and $\xi\eta$, we obtain

$$\mu \sin\phi = \mu' \sin\phi',$$

$$\left.\begin{aligned} \mu'a'\cos^2\phi' - \mu a\cos^2\phi &= (\mu'\cos\phi' - \mu\cos\phi)\,R \\ \mu'b' - \mu b &= (\mu'\cos\phi' - \mu\cos\phi)\,T \\ \mu'h'\cos\phi' - \mu h\cos\phi &= (\mu'\cos\phi' - \mu\cos\phi)\,S \end{aligned}\right\} \ldots \text{(A)}.$$

From these equations we can determine the principal planes and the focal lines of the refracted pencil as follows.

Turn the axes $x'y$ about the axis of z', the axis of the refracted pencil, from x' towards y through a positive acute angle θ, so that the coefficient of the product term in the equation of the orthotomic surface vanishes. This equation will then take the form

$$z' = X^2/2v_1' + Y^2/2v_2',$$

where
$$1/v_1' = a'\cos^2\theta + 2h'\cos\theta\sin\theta + b'\sin^2\theta,$$
$$1/v_2' = a'\sin^2\theta - 2h'\cos\theta\sin\theta + b'\cos^2\theta,$$
$$0 = (a' - b')\sin\theta\cos\theta - h'\cos 2\theta.$$

Hence θ is given by the equation $\tan 2\theta = 2h'/(a' - b')$, which has always a solution for θ between 0 and $\frac{1}{2}\pi$; and

$$1/v_1' + 1/v_2' = a' + b', \quad 1/v_1' - 1/v_2' = (a' - b')\cos 2\theta + 2h'\sin 2\theta$$
$$= \pm\sqrt{(a' - b')^2 + 4h'^2},$$

where the sign of the ambiguity is that of h'.

The focal lines of the refracted pencil are therefore at distances v_1', v_2' in front of the point of incidence, the one which is at distance v_1 being perpendicular to the plane $z'X$, and the angle between this plane and the plane $z'x'$, the plane of refraction, is the positive acute angle θ.

No loss of generality is involved by taking the orthotomic surfaces of both the incident and refracted pencils at the origin. For it is clear from (ii) Art. 171, that if $z' = \frac{1}{2}(a'x'^2 + 2h'x'y + b'y^2)$ be the equation of the second orthotomic surface at the origin, the equation of the orthotomic surface of the refracted pencil, which passes through a point on the axis of the pencil at distance t *beyond* the point of incidence, will be

$$z' + t = \tfrac{1}{2}\,\frac{a'x'^2 + 2h'x'y + b'y^2 + (a'b' - h'^2)(x'^2 + y^2)\,t}{1 + (a' + b')\,t + (a'b' - h'^2)\,t^2}.$$

Making use of this form of the orthotomic surface, we may proceed to determine the constants of a small pencil after successive refractions.

173. *Conditions that the refraction shall not affect the principal planes of the pencil.*

First, in the case of oblique incidence the principal planes of the incident and refracted pencils intersect the plane of incidence in different lines Oz, Oz', the axes of the pencils, and therefore these planes cannot possibly coincide unless they also coincide at the same time with the plane of incidence. In that case h and h' are zero, and therefore S must be zero. Hence the necessary and sufficient conditions that the principal planes may be unaltered are that the plane of incidence be both a principal plane of the incident pencil and also a principal plane of curvature of the refracting surface.

If then the focal lines from which the light diverges are such that the line at distance v_1 is perpendicular to the plane of incidence, and the other at distance v_2 is in the plane of incidence, and if further this plane be a principal plane of the refracting surface, the refracted pencil will diverge from two focal lines perpendicular to and in the plane of incidence respectively, and the distances v_1', v_2' of these lines are given by the standard formulae

$$\left.\begin{aligned}\frac{\mu'\cos^2\phi'}{v_1'} - \frac{\mu\cos^2\phi}{v_1} &= \frac{\mu'\cos\phi' - \mu\cos\phi}{\rho_1}\\ \frac{\mu'}{v_2'} - \frac{\mu}{v_2} &= \frac{\mu'\cos\phi' - \mu\cos\phi}{\rho_2}\end{aligned}\right\}\text{.........(A}'\text{)},$$

where ρ_1, ρ_2 are the principal radii of curvature of the surface, in and perpendicular to the plane of incidence respectively. In each of these formulae the three lengths are algebraic, being of the same sign if measured in the same direction from the point of incidence.

Secondly, for direct incidence, the two orthotomic surfaces of the light and the refracting surface may be at once referred to the same axes, and the relations between the constants of the two surfaces will be

$$\begin{aligned}\mu'a' - \mu a &= (\mu' - \mu)\,R,\\ \mu'b' - \mu b &= (\mu' - \mu)\,T,\\ \mu'h' - \mu h &= (\mu' - \mu)\,S.\end{aligned}$$

We may for convenience choose the principal planes of the incident pencil for the planes of reference; then $h = 0$, and h' will vanish only if S vanish, that is, the planes of the pencil are turned round unless the principal planes of the incident pencil and those of the refracting surface coincide.

174. Thin Astigmatic Lens.

A lens, bounded by any two surfaces, whose principal curvatures are unequal, and having a common normal of those surfaces as axis, is called an *astigmatic* lens. It has the property that a pencil of light diverging from a point near the axis emerges as a pencil with definite focal lines; and conversely, the lens can bring to a focus the rays of an incident pencil with definite focal lines.

Such a lens is used to correct the defect of astigmatism present in some eyes. For the curvatures of the faces of the lens may be chosen so that the focal lines of a pencil after passing the lens are in the positions necessary in order that the crystalline lens of the eye, itself astigmatic, may bring the rays to a focus on the retina.

We can determine the effect of such a lens on any pencil by a double application of the method of Art. 172, as follows.

We treat the lens as thin, and take the common normal to its surfaces as the axis of ζ, and μ as the refractive index of the material of the lens.

Let the principal radii of curvature of the first face of the lens at the origin be ρ_1 and ρ_2, and let its first principal plane of curvature make an angle α with the plane of reference zx or $\zeta\xi$, measured from x towards y. The equation of the first surface is therefore

$$\zeta = \tfrac{1}{2}\left\{\frac{(\xi\cos\alpha + \eta\sin\alpha)^2}{\rho_1} + \frac{(\eta\cos\alpha - \xi\sin\alpha)^2}{\rho_2}\right\}$$
$$= \tfrac{1}{2}(r\xi^2 + 2s\xi\eta + t\eta^2) \quad\text{.................................(i),}$$

where

$$r = \frac{\cos^2\alpha}{\rho_1} + \frac{\sin^2\alpha}{\rho_2}, \quad t = \frac{\sin^2\alpha}{\rho_1} + \frac{\cos^2\alpha}{\rho_2}, \quad s = \sin\alpha\cos\alpha\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right).$$

Let the equation of the second surface of the lens be

$$\zeta = \tfrac{1}{2}(r'\xi^2 + 2s'\xi\eta + t'\eta^2) \quad\text{.................(ii),}$$

where r', s', t' denote quantities similar to r, s and t.

Again, if a pencil of light diverging from two focal lines on the axis at distances v_1 and v_2 from the lens be incident directly, and if the primary focal plane of the pencil make an angle θ with the plane of reference zx, the equation of the orthotomic surface at the origin will be

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2) \quad\text{.................(iii),}$$

where

$$a = \frac{\cos^2\theta}{v_1} + \frac{\sin^2\theta}{v_2}, \quad b = \frac{\sin^2\theta}{v_1} + \frac{\cos^2\theta}{v_2}, \quad h = \sin\theta\cos\theta\left(\frac{1}{v_1} - \frac{1}{v_2}\right).$$

Also let the equation of the orthotomic surface of the light within the lens be

$$z = \tfrac{1}{2}(Ax^2 + 2Hxy + By^2) \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

and of the orthotomic surface at the origin of the emergent light be

$$z = \tfrac{1}{2}(a'x^2 + 2h'xy + b'y^2) \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

where a', h', b' denote quantities similar to a, h, b.

By Malus' theorem we have, as in Art. 172,

$$\zeta - \tfrac{1}{2}(a\xi^2 + 2h\xi\eta + b\eta^2) = \mu\{\zeta - \tfrac{1}{2}(A\xi^2 + 2H\xi\eta + B\eta^2)\},$$

for all points lying on the first surface of the lens.

Hence, substituting from equation (i),

$$\left.\begin{aligned} r - a &= \mu(r - A) \\ t - b &= \mu(t - B) \\ s - h &= \mu(s - H) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

Similarly the constants for the emergent pencil are given by the equations

$$\left.\begin{aligned} r' - a' &= \mu(r' - A) \\ t' - b' &= \mu(t' - B) \\ s' - h' &= \mu(s' - H) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)}.$$

Hence on subtraction we obtain the equations

$$\left.\begin{aligned} a' - a &= (\mu - 1)(r - r') \\ b' - b &= (\mu - 1)(t - t') \\ h' - h &= (\mu - 1)(s - s') \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

By addition and subtraction of the first pair of these, and substitution for the quantities a, b, &c. we obtain the equations

$$\left(\frac{1}{v_1'} + \frac{1}{v_2'}\right) - \left(\frac{1}{v_1} + \frac{1}{v_2}\right) = (\mu - 1)\left\{\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \left(\frac{1}{\rho_1'} + \frac{1}{\rho_2'}\right)\right\},$$

$$\left(\frac{1}{v_1'} - \frac{1}{v_2'}\right)\cos 2\theta' - \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\cos 2\theta = (\mu - 1)\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos 2\alpha - \left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)\cos 2\alpha'\right\},$$

$$\left(\frac{1}{v_1'} - \frac{1}{v_2'}\right)\sin 2\theta' - \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\sin 2\theta = (\mu - 1)\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin 2\alpha - \left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)\sin 2\alpha'\right\} \ldots\ldots\text{(ix)}.$$

Now the quantities on the right of these equations are independent of the incident pencil, moreover the planes of reference zx and zy were taken arbitrarily; let them be chosen so that

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin 2\alpha - \left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)\sin 2\alpha' = 0 \ldots\ldots\ldots\text{(x)}.$$

Since $\alpha' - \alpha$ is the angle between the first principal planes of the two refracting surfaces and is known, this equation gives a single value for $\tan(\alpha + \alpha')$, and therefore these planes of reference occupy a definite position with regard to the lens. They may be found practically by allowing a beam of parallel rays to pass directly through the lens. Since in that case $v_1 = v_2 = \infty$, it follows from equation (ix), the term on the right being zero, that $\sin 2\theta' = 0$; the focal planes of the emergent pencil are therefore the planes of reference so defined. The distances f_1, f_2 of the focal lines of this emergent pencil in front of the lens are given by the equations

$$\frac{1}{f_1} + \frac{1}{f_2} = (\mu - 1)\left\{\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \left(\frac{1}{\rho_1'} + \frac{1}{\rho_2'}\right)\right\},$$

$$\begin{aligned}\frac{1}{f_1} - \frac{1}{f_2} &= (\mu - 1)\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos 2\alpha - \left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)\cos 2\alpha'\right\} \\ &= (\mu - 1)\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2 - 2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)\cos 2(\alpha' - \alpha) \right. \\ &\qquad \left. + \left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)^2\right\}^{\frac{1}{2}} \quad \text{......(xi)},\end{aligned}$$

where, if we suppose α and α' to be positive acute angles, the sign of the square root is that of

$$\sin(\alpha' - \alpha)\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)\right\}.$$

We define these planes as the focal planes of the thin astigmatic lens, and f_1, f_2 as its primary and secondary focal lengths. Equations (ix) for any pencil incident directly will be replaced by

$$\left.\begin{aligned}\left(\frac{1}{v_1'} + \frac{1}{v_2'}\right) - \left(\frac{1}{v_1} + \frac{1}{v_2}\right) &= \frac{1}{f_1} + \frac{1}{f_2} \\ \left(\frac{1}{v_1'} - \frac{1}{v_2'}\right)\cos 2\theta' - \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\cos 2\theta &= \frac{1}{f_1} - \frac{1}{f_2} \\ \left(\frac{1}{v_1'} - \frac{1}{v_2'}\right)\sin 2\theta' - \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\sin 2\theta &= 0\end{aligned}\right\} \text{......(xii)},$$

where θ, θ' are the angles made by the primary planes of the incident and emergent pencils with the primary plane of the lens.

The faces of the lenses are usually worked to cylindrical forms; we should then put ρ_2 and ρ_2' infinite; and the finite radii of curvature ρ_1 and ρ_1' of the faces, and the angle $\alpha' - \alpha$ between the axes of the cylinders may be chosen in an infinite number of ways to obtain requisite focal lengths f_1 and f_2.

175. Refraction through a Prism.

Let a small pencil of light diverging from a point pass through a prism, the axis of the pencil passing in the principal plane of the prism.

Every plane through the axis is a principal plane of the incident light, and every plane through the normal is a principal plane of the refracting plane surface. The plane of incidence and refraction of the axis is therefore a principal plane of the pencil in the prism. If the axis pass in the principal plane of the prism, this plane is also the plane of incidence on the second face, and therefore a principal plane of the emergent light, and we may use the formulae of Art. 173.

(If the axis do not pass in the principal plane, the second plane of incidence does not coincide with the first, which is a principal plane of the light in the prism, and the focal planes of the emergent pencil are turned round.)

If u be the distance of the origin of light from the first point of incidence, and if we take the principal plane of the prism as the primary plane, the distance from the point of incidence of the primary focus of the light in the prism is $\mu u \cos^2\phi' \sec^2\phi$, and that of the secondary focus is μu (A′, Art. 173).

Let t be the length of the path of the axis in the prism; the distances of these points from the point of incidence of the axis on the second face are increased by t; and the distance of the primary focus of the emergent pencil from that point is therefore

$$\frac{\cos^2\psi}{\mu\cos^2\psi'}\left(t + \mu u\,\frac{\cos^2\phi'}{\cos^2\phi}\right),$$

and that of the secondary focus is $\dfrac{t}{\mu} + u$.

The emergent light therefore diverges from two focal lines, whose distance apart is

$$\left(1 - \frac{\cos^2\psi}{\cos^2\psi'}\right)\frac{t}{\mu} \sim \frac{\cos^2\psi\cos^2\phi' - \cos^2\psi'\cos^2\phi}{\cos^2\phi\cos^2\psi'}\,u,$$

or $$(\mu^2 - 1)\tan^2\psi'\,\frac{t}{\mu} \sim (\mu^2-1)\,\frac{\sin^2\phi' - \sin^2\psi'}{\cos^2\phi\cos^2\psi'}\,u.$$

When the axis passes with minimum deviation, $\phi' = \psi'$, and the distance between the focal lines is proportional to t; hence rays passing near the edge give a good image of a small object seen through the prism (cf. Art. 24).

176. Refraction at a Spherical Surface of a pencil diverging from a point.

The orthotomic surface of the pencil at the point of incidence is a sphere, having its centre at the origin of light. Hence the plane of incidence is a principal plane, both of the incident pencil and of the refracting surface. It is therefore also one of the principal planes of the refracted pencil, and is usually taken as the primary plane. The distance of the origin of light from the point of incidence being u, the distances v_1, v_2 of the primary and secondary foci are given by the equations (A′, Art. 173)

$$\frac{\mu' \cos^2 \phi'}{v_1} - \frac{\mu \cos^2 \phi}{u} = \frac{\mu' \cos \phi' - \mu \cos \phi}{\rho},$$

$$\frac{\mu'}{v_2} - \frac{\mu}{u} = \frac{\mu' \cos \phi' - \mu \cos \phi}{\rho}.$$

177. Geometrical Relation between Primary Foci.

When the axis of a pencil diverging from a point is incident on a refracting spherical surface in a given direction at a given point, the line joining any positions of the origin of light and the primary focus of the refracted pencil passes through a fixed point.

Let A be the point of incidence of the axis, BA the fixed direction of the

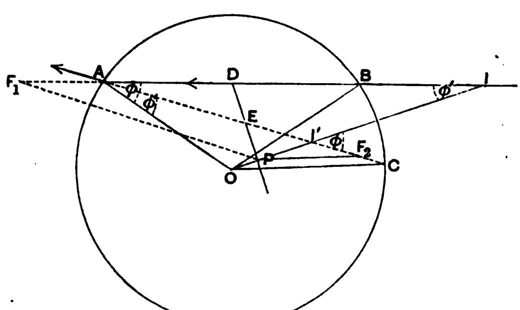


Fig. 85.

axis, CA that of the axis of the refracted pencil.

Since we have $\dfrac{\mu' \cos^2 \phi'}{v_1} - \dfrac{\mu \cos^2 \phi}{u} = \dfrac{\mu' \cos \phi' - \mu \cos \phi}{\rho}$, the straight line which makes intercepts u and v_1 on AB, AC respectively will pass through a fixed point whose coordinates, referred to AB, AC as axes, are

$$-\frac{\mu \cos^2 \phi}{\mu' \cos \phi' - \mu \cos \phi}\,\rho \text{ and } \frac{\mu' \cos^2 \phi'}{\mu' \cos \phi' - \mu \cos \phi}\,\rho.$$

We can construct this point geometrically as follows.

First, the equation is satisfied by $u=\rho\cos\phi$, $v_1=\rho\cos\phi'$, *i.e.* the middle points D, E, of AB and AC are conjugate primary foci.

Secondly, we can take the pair of aplanatic points I and I', which are conjugate to each other for pencils of any magnitude (Art. 41), to lie on AB and AC respectively. They will lie on a radius through O, and by similar triangles the angle AIO is ϕ', the angle $AI'O$ is ϕ. Hence the angles BOI, COI' are both $\phi-\phi'$; therefore OI bisects the arc BC, and is perpendicular to DE. The fixed point is therefore the foot of the perpendicular from O on DE. If through this point P we draw PF_1 parallel to CA to meet AB in F_1, and PF_2 parallel to AB to meet AC in F_2, we may call F_1 and F_2 the first and second principal primary foci, and the positions of any origin of light Q on AB and its primary focus Q_1 on AC will be connected by the equation

$$F_1Q \,.\, F_2Q_1 = -\frac{\mu\mu'\cos^2\phi\cos^2\phi'}{(\mu'\cos\phi'-\mu\cos\phi)^2}\rho^2.$$

In the same way the form of the equation for the secondary focus shews that the line joining the origin to the secondary focus always passes through a fixed point. But this point is here the centre of curvature O. For $u=\rho\sec\phi$, $v_2=\rho\sec\phi'$ are consistent values; *i.e.* the points in which a perpendicular through O to the radius OA cuts the axes of the pencils are conjugate secondary foci. Also I and I' are always conjugate, and they lie on a radius, so that the line joining any two conjugate secondary foci always passes through the centre.

178. Spherical Refracting Surface.

The existence of the focal lines and the formulae for their position can also be proved, in the case of a spherical surface, by simple geometrical methods.

If rays diverging from a point Q be reflected (or refracted) at a spherical surface, centre O, they will then touch a surface of revolution about QO as axis. The rays reflected at a small part BC of any meridian will all pass through the subtense drawn to the caustic curve from the intersection of the rays Bb, Cc. This subtense is proportional to the square of the arc bc, and hence to the square of BC. If we neglect this lateral aberration as being proportional to the square of the breadth of a small incident pencil, we may say that all the rays from BC pass through the point q_1, the point in which the axis Aq_1q_2 touches the caustic. Rotating the figure about QO, we see that all the rays reflected

in the neighbourhood of A pass through a small circular arc through q_1 perpendicular to the plane QAO. To our order of approximation this circular arc is indistinguishable from a straight

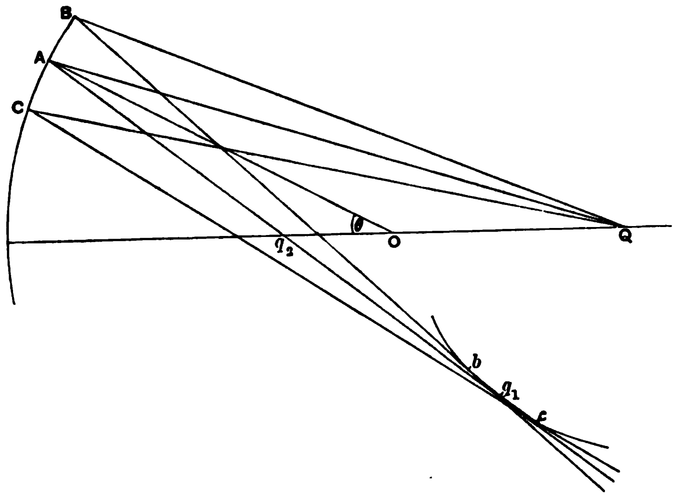


Fig. 86.

line. The plane of incidence of the axis is the primary focal plane, the point q_1 the primary focus, and this small straight line the primary focal line.

Again, we see that all the reflected rays pass accurately through the part of the axis QO terminated by the points in which the extreme rays meet it.

But this line bears no simple relation to the refracted pencil. The section of the pencil, however, by a plane through q_2 perpendicular to its axis is an elongated figure, the maximum breadth of which perpendicular to the primary plane is clearly proportional to the square of the breadth of the pencil near A. This figure must therefore be regarded as a straight line, lying in the primary plane, and passing through q_2, the secondary focus.

The above arguments assume throughout that the distances AQ, Aq_1, Aq_2 are great compared with the linear dimensions of the pencil near A.

179. Formulae for Primary and Secondary Foci.

I. *Reflection.*

If ϕ be the angle of incidence, θ the supplement of the angle AOQ, the angles made by AQ and Aq_2 with the axis QO are respectively $\theta-\phi$ and $\theta+\phi$; and if $AQ=u$, $Aq_1=v_1$, $AO=\rho$,

$$\text{the arc } AB=\rho d\theta=u\sec\phi\, d(\theta-\phi)=v_1\sec\phi\, d(\theta+\phi).$$

Hence
$$\frac{d\theta}{d\phi}=1-\frac{\rho\cos\phi}{u}=\frac{\rho\cos\phi}{v_1}-1,$$

i.e.
$$\frac{1}{v_1}+\frac{1}{u}=\frac{2}{\rho\cos\phi}.$$

It follows that if the point A and the direction of QA be fixed, the line joining any origin of light Q and its primary focus passes through the point where AO is cut by the line joining the middle points of the chords intercepted by the circle on the axes of the incident and reflected pencils (cf. Art. 177).

If $Aq_2=v_2$, we have by expressing that the area of the triangle q_2AQ is the sum of the areas of the triangles q_2AO and AOQ,

$$v_2u\sin 2\phi=v_2\rho\sin\phi+u\rho\sin\phi,$$

i.e.
$$\frac{1}{v_2}+\frac{1}{u}=\frac{2\cos\phi}{\rho}.$$

II. *Refraction.*

With the same notation as in the case of reflection, we have the angles AQO and Aq_2O respectively equal to $\theta-\phi$ and $\theta-\phi'$, where ϕ' is the angle of refraction.

Hence the arc $AB=\rho d\theta=u\sec\phi\, d(\theta-\phi)=v_1\sec\phi'\, d(\theta-\phi')$.

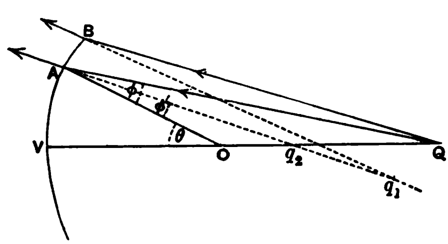


Fig. 87.

But since $\mu\sin\phi=\mu'\sin\phi'$, we have

$$\mu\cos\phi\, d\phi=\mu'\cos\phi'\, d\phi',$$

or
$$\mu\cos\phi\left(1-\frac{\rho\cos\phi}{u}\right)=\mu'\cos\phi'\left(1-\frac{\rho\cos\phi'}{v_1}\right),$$

i.e.
$$\frac{\mu'\cos^2\phi'}{v_1}-\frac{\mu\cos^2\phi}{u}=\frac{\mu'\cos\phi'-\mu\cos\phi}{\rho}.$$

For the secondary focus q_2, the same condition as before gives

$$\rho u \sin\phi = \rho v_2 \sin\phi' + u v_2 \sin(\phi - \phi'),$$

i.e.
$$\frac{\sin\phi}{v_2} - \frac{\sin\phi'}{u} = \frac{\sin(\phi-\phi')}{\rho},$$

or
$$\frac{\mu'}{v_2} - \frac{\mu}{u} = \frac{\mu'\cos\phi' - \mu\cos\phi}{\rho}.$$

If we denote OQ by p, Oq_2 by q_2, OA by r where $r = -\rho$, then by projecting AQ on AO it follows that

$$u\cos\phi = p\cos\theta - r; \quad v_2\cos\phi' = q_2\cos\theta - r.$$

Hence
$$\frac{\mu'\cos\phi'}{q_2\cos\theta - r} - \frac{\mu\cos\phi}{p\cos\theta - r} = \frac{\mu'\cos\phi' - \mu\cos\phi}{-r},$$

i.e.
$$\mu'\cos\phi'\frac{q_2}{q_2\cos\theta - r} = \mu\cos\phi\frac{p}{p\cos\theta - r},$$

which gives the equation

$$\frac{1}{\mu' q_2\cos\phi'} - \frac{1}{\mu p\cos\phi} = \left(\frac{1}{\mu'\cos\phi'} - \frac{1}{\mu\cos\phi}\right)\frac{\cos\theta}{r}.$$

180. Boundary of any small pencil.

The rays of any small pencil are determined by the fact that they intersect the two focal lines. If the area in which they meet any orthotomic surface be known, the general shape of the pencil may be found as follows.

Let the extreme rays of the small pencil cut its orthotomic surface at the origin in an ellipse of semi-axes a and b, *lying in the principal planes of the pencil.* The equations of this ellipse may be written

$$x^2/a^2 + y^2/b^2 = 1, \quad z = 0.$$

The equations of the focal lines are $x = 0$, $z = v_1$, and $y = 0$, $z = v_2$; and the equations of any straight line meeting both are

$$x + \lambda(z - v_1) = 0, \quad y + \mu(z - v_2) = 0.$$

If this ray pass through the boundary of the ellipse, we have

$$\lambda^2 v_1^2/a^2 + \mu^2 v_2^2/b^2 = 1;$$

and therefore, on substituting for λ and μ, the extreme rays generate the quartic surface

$$\frac{x^2}{a^2(1 - z/v_1)^2} + \frac{y^2}{b^2(1 - z/v_2)^2} = 1.$$

The section of this surface by any plane perpendicular to the axis of the pencil is an ellipse, and two such sections will be circular. The positions of these sections are given by the equation

$$a^2 (1 - z/v_1)^2 = b^2 (1 - z/v_2)^2.$$

If v_1 and v_2 have the same sign, the position of the smaller circle of the two is given by $(a + b)/z = a/v_1 + b/v_2$, and its radius is $ab (v_1 \sim v_2)/(av_2 + bv_1)$. This circle lies between the focal lines and is called the *Least Circle of Confusion.*

If v_1 and v_2 have opposite signs, the position of the smaller circle, which is the one between the focal lines, is given by $(a - b)/z = a/v_1 - b/v_2$, and its radius is

$$ab (v_2 - v_1)/(av_2 - bv_1).$$

It must be noticed that there are circular sections only when the axes of the elliptic section at the origin lie in the principal planes of the pencil. If the equations of the ellipse be

$$z = 0, \quad ax^2 + 2hxy + by^2 = 1,$$

the extreme rays generate the surface

$$\frac{ax^2}{\left(1 - \dfrac{z}{v_1}\right)^2} + \frac{2hxy}{\left(1 - \dfrac{z}{v_1}\right)\left(1 - \dfrac{z}{v_2}\right)} + \frac{by^2}{\left(1 - \dfrac{z}{v_2}\right)^2} = 1,$$

and none of the sections of this surface by planes perpendicular to the axis are circular.

It is usual to consider that the image of a finite object, formed by oblique reflections or refractions, is composed of the circles of least confusion answering to each point of the object. For if the pencils of light be received on a screen, each point of the object will be represented there by a small straight line, if the screen be at either principal focus, or by an ellipse, if the screen be in any other position. The image formed by the focal lines may be very much blurred, as for each point they may run across the general direction of the image. Taking all directions into account, the circles of least confusion on the whole give the most distinct image.

The circle of least confusion is not the section of the pencil which is least in area; this is an ellipse lying midway between the focal lines; but the possible variations in the directions of its axes forbid our taking it to determine the image.

181. Example.

A small circular disc is seen in a plane mirror consisting of a plate of glass silvered at the back, and the disc is at right angles to the initial course of the central ray to the eye. Prove that it appears an ellipse in which the ratio of the axes is

$$\mu s \cos^2\phi' + s' \cos^2\phi : (\mu s + s') \cos^2\phi',$$

where s and s' are the lengths of the course of the central ray without and within the glass respectively.

Let $DRQPE$ be the course of the central ray entering the eye E; this will be in one plane throughout, and if we retrace the course of a small pencil of rays from E, passing through the edge of the disc, we may regard this plane as the primary focal plane throughout, and apply the formulae of Art. 173.

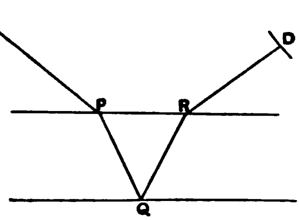


Fig. 88.

Let $EP=u$, $PQ=\frac{1}{2}s'$, $RD=v$; let ϕ, ϕ' be the angles of incidence and refraction.

Let v_1, v_2 be the distances of the primary and secondary foci from P after refraction at P; $v_1=\mu u \cos^2\phi' \sec^2\phi$, $v_2=\mu u$.

The reflection at a plane being perfect, the distances of these foci from Q are unaltered by the reflection there, and therefore, if v_1', v_2' be the distances from R of the foci before incidence at R, $v_1'=s'+\mu u \cos^2\phi' \sec^2\phi$, $v_2'=s'+\mu u$.

Let V_1, V_2 be the distances from R of the foci of the emergent pencil,

$$V_1=\frac{v_1'\cos^2\phi}{\mu\cos^2\phi'}=u+\frac{s'\cos^2\phi}{\mu\cos^2\phi'}, \qquad V_2=\frac{v_2'}{\mu}=u+\frac{s'}{\mu}.$$

These distances are measured from R in the direction of DR produced.

Again, suppose that just before incidence at P the rays pass orthogonally through an ellipse of semi-axes a and b in the primary and secondary planes respectively; they therefore meet the plane mirror at P in an ellipse of semi-axes $a\sec\phi$ and b, and meet their orthotomic surface on leaving P in an ellipse of semi-axes $a\sec\phi\cos\phi'$ and b. They will meet their orthotomic surface at Q in an ellipse of semi-axes $a\sec\phi\cos\phi'(1+\frac{1}{2}s'/v_1)$ and $b(1+\frac{1}{2}s'/v_2)$. The reflection at Q will not alter the shape of the pencil, and the rays cross their orthotomic surface just before incidence at R in an ellipse of semi-axes $a\sec\phi\cos\phi'(1+\frac{1}{2}s'/v_1)\{1+\frac{1}{2}s'/(v_1+\frac{1}{2}s')\}$ and $b(1+\frac{1}{2}s'/v_2)\{1+\frac{1}{2}s'/(v_2+\frac{1}{2}s')\}$, or $a\sec\phi\cos\phi' v_1'/v_1$ and bv_2'/v_2.

By the refraction at R the axes are altered in the inverse ratio to the change at P, and therefore the emergent rays meet their orthotomic surface at R in an ellipse of semi-axes av_1'/v_1 and bv_2'/v_2.

By Art. 180, the section of the emergent pencil by the plane at D, lying outside the focal lines and in the negative direction of the axis, is a circle if

$$\frac{av_1'/v_1 - bv_2'/v_2}{-v} = \frac{av_1'/v_1}{V_1} - \frac{bv_2'/v_2}{V_2},$$

i.e. if
$$a\left(\frac{1}{v}+\frac{1}{V_1}\right)\frac{v_1'}{v_1} = b\left(\frac{1}{v}+\frac{1}{V_2}\right)\frac{v_2'}{v_2},$$

or
$$\frac{b}{a} = \frac{V_1+v}{V_2+v}\cdot\frac{v_1'}{V_1}\cdot\frac{V_2}{v_2'}\cdot\frac{v_2}{v_1} = \frac{V_1+v}{V_2+v}$$
$$= \frac{u+v+s'\cos^2\phi/\mu\cos^2\phi'}{u+v+s'/\mu} = \frac{\mu s\cos^2\phi'+s'\cos^2\phi}{(\mu s+s')\cos^2\phi'}.$$

The rays from the disc therefore enter the eye in the shape of a small cone, any cross-section of which is an ellipse, whose axes are in the required ratio.

EXAMPLES.

1. A luminous point is placed in front of a thick plate of glass with parallel faces; shew that the rays emerging after any number of internal reflections are normals to two series of equal and similarly placed prolate quadrics of revolution, each of which has one focus coincident with the successive images of the point due to the faces of the plate considered as plane mirrors situated in air.

2. A small pencil diverging from a point falls obliquely on a looking-glass, and emerges after reflection at the silvered back. Shew that it proceeds from two focal lines, whose distance from one another is

$$\frac{2t}{\mu}(\mu^2-1)\tan^2\phi'\sec\phi',$$

where ϕ' is the angle of refraction, and t the thickness of the glass.

3. A small pencil of rays diverging from a point P, whose distance is variable, is incident on a refracting sphere at a given point in a given direction; if Q be the primary focus after refraction through the sphere, F_2 the position of Q when the incident pencil consists of parallel rays, F_1 that of P when the emergent pencil consists of parallel rays, prove that

$$PF_1 . F_2Q = \frac{a^2}{16}\frac{\sin^2 2\phi\cos^2\phi'}{\sin^2(\phi-\phi')},$$

where a is the radius of the sphere, ϕ the angle of incidence on the sphere, ϕ' the angle of refraction.

4. The surfaces of a double-concave lens are spheres of radii r and s, and its thickness is t. Light is incident at angle ϕ at a given point on the axis from a point Q lying in a given direction. If Q' be the primary focus, and F, F' the positions of Q, Q', when Q', Q respectively are at an infinite distance, then

$$QF . Q'F' = -\cos^2\phi\cos^2\psi\sec^2\phi'\sec^2\psi'[\rho\sigma R+\rho+\sigma]^{-2},$$

where
$$\mu(s+t)\operatorname{cosec}\psi = (s+t)\operatorname{cosec}\psi' = s\operatorname{cosec}\phi' = \mu s\operatorname{cosec}\phi,$$
$$\rho = \sec^2\phi'(\mu\cos\phi'-\cos\phi)/r, \quad \sigma = \sec^2\psi'(\mu\cos\psi'-\cos\psi)/s,$$

and $R\mu$ is the positive root of

$$x^2+2x(s+t)\cos\phi'+t(2s+t)=0.$$

5. A small pencil, diverging from a point at distance u in the plane of zx, is incident at the origin on the reflecting surface $2z = rx^2 + 2sxy + ty^2$, the angle of incidence being ϕ. Prove that the distances from the origin of the focal lines of the reflected pencil are the roots of the equation

$$(1/v + 1/u - 2r \sec\phi)(1/v + 1/u - 2t \cos\phi) = 4s^2.$$

6. Prove that the principal curvatures of a small mirror can be so chosen that it will reflect to a point all pencils incident on it from points which lie in a given direction with respect to the mirror. Prove that then the principal planes of any pencil incident in this direction will make the same angle with the plane of incidence before and after reflection; and that the reciprocals of the distances of the focal lines from the point of incidence will be increased by the same amount for all such pencils.

7. The front of a thin lens is plane, the back is cylindrical of radius r and is silvered. A small pencil diverging from a point is incident centrically at angle ϕ in a plane making an angle α with the axis of the cylinder. Shew that the distances from the lens of the foci of the emergent pencil are $-u$ and

$$\left[\frac{2\mu}{r}\cos\phi'(1 + \sin^2\alpha \tan^2\phi) - \frac{1}{u}\right]^{-1}.$$

8. A pencil of light diverging from a point at distance u from a thin lens passes through it directly. The front surface of the lens is of principal curvatures $1/\rho_1$, $1/\rho_2$, and the second surface is the same surface, turned about the common normal through an angle α; shew that the distances from the lens of the focal lines are given by

$$1/v_1 - 1/u = 1/u - 1/v_2 = (\mu - 1)(1/\rho_1 - 1/\rho_2)\sin\alpha.$$

9. A thin astigmatic lens refracts a pencil of parallel rays so that they diverge from focal lines at distances f_1, f_2 from the lens. Prove that they may be made to diverge from a point at distance f by placing close to the lens another coaxial lens suitably adjusted with one face plane.

Prove also that if the correcting lens be turned about the axis through an angle θ, the distances from either lens of the focal lines of a pencil, which before refraction diverged from a point on the axis at distance u, are given by

$$1/v_1 + 1/v_2 = 2(1/u + 1/f); \quad 1/v_1 - 1/v_2 = 2(1/f_1 - 1/f_2)\sin\theta.$$

10. A small pencil is incident directly on a thin astigmatic lens; shew that the principal planes of the pencil remain the same on emergence from the lens, if the angle between a principal plane of the pencil and a principal plane of the first surface be

$$\tfrac{1}{2}\cot^{-1}\left(\cot 2\beta - \frac{\delta_1}{\delta_2}\operatorname{cosec} 2\beta\right),$$

where β is the angle between a pair of principal planes of the two surfaces, and δ_1, δ_2 are the differences of the principal curvatures of the first and second surfaces respectively.

11. Two thin astigmatic lenses are placed on an axis at distance t apart; the principal focal lengths of each lens are $+f$, $-f$, and they are arranged so that the angle between corresponding principal planes of the two lenses is $\frac{1}{4}\pi$. A small pencil is directly incident on the system from a point at distance af in front of one lens; prove that the focal lines of the emergent pencil are at distances $\beta_1 f$, $\beta_2 f$ beyond the other lens, where β_1, β_2 are the roots of the quadratic

$$(atx)^2 - x^2(t+af)^2 - a^2(t+xf)^2 + \{t+(x+a)f\}^2 = 0.$$

12. A narrow beam of light is incident centrically on a thin lens at angle ϕ, the rays at incidence being normal to $z = \frac{1}{2}(ax^2 + 2hxy + by^2)$. Shew that after passing through the lens they are normal to

$$z = \tfrac{1}{2}\{(a+\lambda)x^2 + 2hxy + (b+\lambda')y^2\},$$

where $\lambda' = (\mu\cos\phi' - \cos\phi)(r^{-1} - s^{-1})$, and $\lambda = \lambda'\sec^2\phi$,

r and s being the radii of the surfaces, supposed spherical. Shew that the focal lines are rotated through an angle

$$\tfrac{1}{2}\cot^{-1}\left\{\frac{a-b}{2h} + \frac{(a-b)^2 + 4h^2}{2h(\lambda - \lambda')}\right\}.$$

13. A small pencil of white light diverges from a point at distance u from the edge of a prism of angle i, and the mean ray of index of refraction μ passes near the edge with minimum deviation.

Prove that on emergence the pencil formed by coloured rays of index $\mu + \partial\mu$ diverges from focal lines separated by a distance

$$4u\frac{\partial\mu}{\mu}\frac{(\mu^2-1)\tan^2\frac{i}{2}}{1-\mu^2\sin^2\frac{i}{2}}.$$

14. A small pencil of rays passes through a prism, the axis passing with minimum deviation in the principal plane. The distances of the primary and secondary foci of the incident pencil from the point of incidence are v_1 and v_2, and one principal plane of the pencil makes an angle θ with the plane of incidence. The length of the path in the prism is t. Prove that if

$$\cos 2\theta = \frac{\mu^2-1}{\mu}\frac{t}{v_1 \sim v_2}\tan^2\frac{i}{2},$$

the focal planes of the emergent pencil will make equal angles with the principal plane of the prism.

15. A pencil of light is incident on a prism of refracting angle i near its edge, the angles of incidence and refraction at the first face being ϕ and ϕ', and the plane of incidence of the axis makes an angle ω with the principal plane of the prism. The orthotomic surface at the point of incidence referred to the axis of the pencil as axis of z, and the plane of incidence as plane of zx, is approximately

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2).$$

Prove that on emergence the sections of the new orthotomic surfaces by the plane of emergence are principal sections of those surfaces if

$$2h\cos\phi\cos\phi' = \tan 2\theta\,(a\cos^2\phi \sim b\cos^2\phi'),$$

where $\sin\theta = \mu\sin\omega\sin i\operatorname{cosec}\psi$, and ψ is the angle of emergence from the prism.

16. The curvatures of a refracting surface are equal and opposite, of magnitude ρ; the plane of incidence of the axis of a small pencil bisects the angle between the planes of principal curvature. The orthotomic surfaces near the origin referred to the axes of the incident and refracted pencils as axes of z and z', and the plane of incidence as planes of zx and $z'x'$, are

$$2z = ax^2 + 2hxy + by^2 \text{ and } 2z' = a'x'^2 + 2h'x'y' + b'y'^2$$

respectively. Shew that for all angles of incidence

$$\mu'b' = \mu b, \quad \left(\frac{h-\rho}{h'-\rho}\right)^2 = \frac{\mu' a}{\mu a'}.$$

17. A small pencil passes obliquely through a thin astigmatic lens, its axis passing without deviation. Shew that the relations between the distances of the focal lines of the emergent and incident pencils from the lens, and between the principal planes, are given by the equations

$$\left(\frac{1}{v_1'} + \frac{1}{v_2'}\right) - \left(\frac{1}{v_1} + \frac{1}{v_2}\right)$$
$$= \frac{\mu \cos\phi' - \cos\phi}{\mu - 1}\left\{\sec^2\phi\left(\frac{\cos^2\epsilon}{f_1} + \frac{\sin^2\epsilon}{f_2}\right) + \frac{\sin^2\epsilon}{f_1} + \frac{\cos^2\epsilon}{f_2}\right\},$$

$$\left(\frac{1}{v_1'} - \frac{1}{v_2'}\right)\cos 2\theta' - \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\cos 2\theta$$
$$= \frac{\mu \cos\phi' - \cos\phi}{\mu - 1}\left\{\sec^2\phi\left(\frac{\cos^2\epsilon}{f_1} + \frac{\sin^2\epsilon}{f_2}\right) - \frac{\sin^2\epsilon}{f_1} - \frac{\cos^2\epsilon}{f_2}\right\},$$

$$\left(\frac{1}{v_1'} - \frac{1}{v_2'}\right)\sin 2\theta' - \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\sin 2\theta$$
$$= \frac{\mu \cos\phi' - \cos\phi}{\mu - 1}\left(\frac{1}{f_1} - \frac{1}{f_2}\right)\sin 2\epsilon \sec\phi,$$

where θ and θ' are the angles between the first principal planes of the pencil and the plane of incidence of its axis, and ϵ is the angle between the first principal plane of the lens and the plane of incidence.

18. Light diverging in the form of a small right circular cone from a point at distance u from the point of incidence of the axis is refracted at a spherical surface. Shew that the distance v of the centre of the circle of least confusion is given by

$$\frac{\mu' \cos\phi'}{v} - \frac{\mu \cos\phi}{u} = \frac{1 + \cos\phi\cos\phi'}{\cos\phi + \cos\phi'}\,\frac{\mu'\cos\phi' - \mu\cos\phi}{\rho}.$$

19. A small pencil of rays diverging from a point in the form of a right circular cone is refracted through a system of plates of thicknesses $t_1, t_2 \ldots t_n$, and emerges into air again. Shew that the least circle of confusion of the emergent pencil is midway between its focal lines, at a distance from the point of emergence equal to

$$u + \tfrac{1}{2}\cos^2\phi_0 \operatorname{cosec}\phi_0 \sum_1^n t_r \sin\phi_r \sec^3\phi_r + \tfrac{1}{2}\operatorname{cosec}\phi_0 \sum_1^n t_r \tan\phi_r,$$

where u is the distance of the origin of light from the first point of incidence, ϕ_0 the initial angle of incidence, ϕ_r the angle of incidence in the rth plate.

20. A pencil in the form of a small right circular cylinder falls on a double concave lens of axial thickness t, and the axis of the pencil passes through the lens without deviation; shew that after emergence the distances of the focal lines and of the centre of the circle of least confusion from the point of emergence are respectively

$$\frac{\cos^2\phi\,(\kappa_1 x+\cos^2\phi')}{\kappa_1\kappa_2 x+(\kappa_1+\kappa_2)\cos^2\phi'},\qquad \frac{\kappa_1 x+1}{\kappa_1\kappa_2 x+\kappa_1+\kappa_2},$$

and
$$\frac{2+\kappa_1 x\,(1+\sec^2\phi')}{(\kappa_1+\kappa_2)\,(1+\sec^2\phi)+\kappa_1\kappa_2 x\,(1+\sec^2\phi\sec^2\phi')};$$

where
$$\mu x=\sqrt{(r+s+t)^2-(r+s)^2\sin^2\phi'}-(r+s)\cos\phi',$$

and
$$\kappa_1=\frac{\mu\cos\phi'-\cos\phi}{r},\qquad \kappa_2=\frac{\mu\cos\phi'-\cos\phi}{s}.$$

21. Prove that two equal eyes viewing each other by oblique reflection in a convex spherical mirror of radius r will appear to one another as equal and similar ellipses of eccentricity

$$\frac{2\sin\phi\,\{uv\,(u+v)\,r\cos\phi+u^2v^2\,(1+\cos^2\phi)\}^{\frac{1}{2}}}{(u+v)\,r\cos\phi+2uv},$$

where u and v are the distances of the eyes from the reflecting area, and ϕ is the obliquity.

CHAPTER XI.

CHARACTERISTIC FUNCTION.

182. BY Malus' theorem a system of rays once orthogonal to a surface retains that property after any refractions; and any orthotomic surface is such that the reduced path of a ray from the original orthotomic surface to this surface is constant for all points on it. Hence it follows that, if the direction-cosines (l, m, n) of any ray of such a pencil be expressed in terms of the coordinates (x, y, z) of a point on that ray, l, m, n must be proportional to the differential coefficients of a single-valued function of the coordinates; and that the reduced path may be taken as that function.

Let the reduced path from the original orthotomic surface to a point (x, y, z) be expressed in terms of only the coordinates of that point and the constants of the original or any other orthotomic surface, and let V denote the function so obtained. V is called the *characteristic function*; and the direction-cosines of the ray are given by the equations

$$\mu l = \frac{\partial V}{\partial x}, \quad \mu m = \frac{\partial V}{\partial y}, \quad \mu n = \frac{\partial V}{\partial z}.$$

The following proof will hold, whether the point P, of coordinates (x, y, z), lie in the same medium as the original orthotomic surface, or whether the light has been refracted at various surfaces, or has passed through a heterogeneous medium.

Let the path of the ray that passes through P be AP, where A is a point on the original orthotomic surface, and let an orthotomic surface of the rays be drawn through P. If P be displaced arbitrarily to P', and the adjacent ray $A'QP$ cut the orthotomic surface in Q, it follows, since the reduced path is the same for the rays AP and $A'Q'$, that

$$\begin{aligned} dV = \mu P'Q &= \mu \,(\text{projection of } PP' \text{ on the normal}) \\ &= \mu\,(ldx + mdy + ndz). \end{aligned}$$

Hence $$\frac{\partial V}{\partial x} = \mu l, \quad \frac{\partial V}{\partial y} = \mu m, \quad \frac{\partial V}{\partial z} = \mu n;$$

and therefore in all cases V satisfies the differential equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \mu^2.$$

When a pencil is refracted from one homogeneous medium into another at a known surface, the form of the characteristic function will change; and if V, V' be the two forms for points in the two media respectively, $V = V'$ at all points of the surface of separation. Hence if the forms of V and V' be known, the constant coefficients that occur in their equations can be connected. This is the method we have followed in the last chapter, using the first approximations to V in the neighbourhood of the point of incidence.

The introduction of the characteristic function is due to Sir W. R. Hamilton*; theoretically all properties of an orthogonal pencil can be deduced from the existence of this function.

183. The difficulty which practically arises in dealing with the characteristic function is that, given the equation of a surface, it is not possible analytically to express the normal distance r of a point from it as an explicit function of the coordinates (x, y, z) of that point. Except in the case where the surface is a plane, the number of normals which can be drawn from a point to a surface will exceed unity, and the equation connecting r with the coordinates of the point will not give a single-valued form for r. Thus the equation of the surface parallel to an ellipsoid involves r^2 to the sixth degree.

But as a rule the normals are confined to a certain part of the given orthotomic surface, and we may approximate to the corresponding value of r to any required degree of accuracy. Thus it has been shewn in Art. 171 that, if the orthotomic surface in the immediate neighbourhood of the origin can be written in the form $z = \frac{1}{2}(x^2/v_1 + y^2/v_2)$, the characteristic function for a point in the immediate neighbourhood of the axis of z is

$$V_0 - \mu\left\{z - \frac{1}{2}\left(\frac{x^2}{v_1 - z} + \frac{y^2}{v_2 - z}\right)\right\}$$

to the same degree of accuracy.

Another illustration is given in the next Article.

In a similar manner, if we try to deduce the paths of rays in heterogeneous media by solving the differential equation satisfied by V, it is necessary to choose the form of solution in such a way that for definite values of the arbitrary constants involved, it may agree with the form of the original orthotomic surfaces. In the general case this difficulty proves insuperable.

* Hamilton, *Trans. Roy. Irish Acad.* Vol. xv., 1828, Vol. xvi., 1831.

184. *The orthotomic surface at the origin of a small pencil is given to the third order as*

$$z=\tfrac{1}{2}(ax^2+2hxy+by^2)+\tfrac{1}{3}(cx^3+3dx^2y+3exy^2+fy^3),$$

to find the characteristic function for a point near *the origin to the same order.*

The equations of the normal to this surface at (x, y, z) are

$$\frac{\xi-x}{ax+hy+\dots}=\frac{\eta-y}{hx+by+\dots}=\frac{\zeta-z}{-1}=\frac{-r}{\{1+(ax+hy)^2+(hx+by)^2\}^{\frac{1}{2}}}\dots\text{(i)}.$$

If we confine ourselves to points near the origin, ξ, η, ζ, and r are all small quantities of at least the first order; therefore ξ differs from x, η from y, by terms of the second order.

We have, correct to terms of the third order,

$$\begin{aligned}\zeta&=z+r\{1-\tfrac{1}{2}(ax+hy)^2-\tfrac{1}{2}(hx+by)^2\}\\&=r+\tfrac{1}{2}(ax+hy)\{x-r(ax+hy)\}+\tfrac{1}{2}(hx+by)\{y-r(hx+by)\}\\&\quad+\tfrac{1}{3}(cx^3+3dx^2y+3exy^2+fy^3)\dots\dots\text{(ii)}.\end{aligned}$$

Also to terms of the second order

$$\xi=x-r(ax+hy),\quad \eta=y-r(hx+by)\dots\dots\text{(iii)},$$

and therefore $\qquad x=\xi+r(a\xi+h\eta),\quad y=\eta+r(h\xi+b\eta)\dots\dots\text{(iv)}.$

Substitute from (iii) and (iv) in (ii),

$$\begin{aligned}\zeta&=r+\tfrac{1}{2}(ax+hy)\,\xi+\tfrac{1}{2}(hx+by)\,\eta+\tfrac{1}{3}(cx^3+3dx^2y+3exy^2+fy^3)\\&=r+\tfrac{1}{2}[(a\xi^2+2h\xi\eta+b\eta^2)+r\{(a^2+h^2)\xi^2+2(a+b)h\xi\eta+(b^2+h^2)\eta^2\}]\\&\quad+\tfrac{1}{3}(c\xi^3+3d\xi^2\eta+3e\xi\eta^2+f\eta^3)\dots\dots\text{(v)}.\end{aligned}$$

In the bracket we may put $r=\zeta$; hence to the third order of small quantities

$$\begin{aligned}r&=\zeta-\tfrac{1}{2}[(a\xi^2+2h\xi\eta+b\eta^2)+\zeta\{(a^2+h^2)\xi^2+2(a+b)h\xi\eta+(b^2+h^2)\eta^2\}]\\&\quad-\tfrac{1}{3}(c\xi^3+3d\xi^2\eta+3e\xi\eta^2+f\eta^3)\dots\dots\text{(vi)}.\end{aligned}$$

The characteristic function V at the point (ξ, η, ζ) is therefore $V_0-\mu r$, where r has the form given in (vi); and it is easy to verify that V satisfies the equation $\left(\frac{\partial V}{\partial\xi}\right)^2+\left(\frac{\partial V}{\partial\eta}\right)^2+\left(\frac{\partial V}{\partial\zeta}\right)^2=\mu^2$, if we neglect the cubes of the coordinates in the squares of the differential coefficients, since these would partly arise from the terms of the fourth order in the equation of the orthotomic surface.

If we confine ourselves only to points near the axis and not necessarily near the origin, *i.e.* if ζ and r be finite quantities, it can be shewn that to the third order of the small quantities ξ and η,

$$\begin{aligned}V=V_0-\mu\,[\zeta&-\tfrac{1}{2}\{(a\xi^2+2h\xi\eta+b\eta^2)-(ab-h^2)\,\zeta(\xi^2+\eta^2)\}/D\\&-\tfrac{1}{3}(cx^3+3dx^2y+3exy^2+fy^3)],\end{aligned}$$

where we must substitute in the last bracket

$$x=\{\xi-(b\xi-h\eta)\,\zeta\}/D,\quad y=\{\eta-(a\eta-h\xi)\,\zeta\}/D,$$
$$D=1-(a+b)\,\zeta+(ab-h^2)\,\zeta^2.$$

This form would be necessary if we wished to determine the orthotomic surface, for example, which meets the axis of z at a given point.

185. Liouville's Theorem. Path of a ray in heterogeneous media.

Let V be any solution of the equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \mu^2,$$

which involves two arbitrary constants α, β by other than simple addition; the surfaces given by the equations

$$\frac{\partial V}{\partial \alpha} = \alpha', \quad \frac{\partial V}{\partial \beta} = \beta',$$

where α', β' are arbitrary constants, intersect in the paths of rays.

Since $V = F(x, y, z, \alpha, \beta)$ satisfies the differential equation for all values of α, the function $F(x, y, z, \alpha + \partial\alpha, \beta)$ also satisfies it. Hence on substitution and subtraction

$$\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x \partial \alpha} + \frac{\partial V}{\partial y}\frac{\partial^2 V}{\partial y \partial \alpha} + \frac{\partial V}{\partial z}\frac{\partial^2 V}{\partial z \partial \alpha} = 0,$$

i.e.

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\frac{\partial V}{\partial \alpha} = 0.$$

Hence the ray lies on the surface

$$\frac{\partial V}{\partial \alpha} = \alpha'.$$

Similarly it lies on the surface

$$\frac{\partial V}{\partial \beta} = \beta'.$$

186. Medium stratified in parallel planes.

Let μ be a function of x only; a solution of the differential equation is obtained by putting

$$\frac{\partial V}{\partial y} = \alpha, \quad \frac{\partial V}{\partial z} = \beta, \quad \frac{\partial V}{\partial x} = (\mu^2 - \alpha^2 - \beta^2)^{\frac{1}{2}},$$

and therefore

$$V = \alpha y + \beta z + \int (\mu^2 - \alpha^2 - \beta^2)^{\frac{1}{2}}\, dx.$$

The surfaces which contain the rays are given by the equations

$$y - \alpha' = \alpha \int (\mu^2 - \alpha^2 - \beta^2)^{-\frac{1}{2}}\, dx,$$

$$z - \beta' = \beta \int (\mu^2 - \alpha^2 - \beta^2)^{-\frac{1}{2}}\, dx,$$

and any ray therefore lies wholly in the plane

$$(y - \alpha')/\alpha = (z - \beta')/\beta.$$

Suppose that the rays diverge from a bright point at the origin of coordinates, and that we take the initial plane of incidence of a ray as the plane of xy. For this ray

$$\frac{\partial V}{\partial y} = \alpha = \mu_0 \sin \phi_0 = \mu \sin \phi,$$

and the equation of the ray is

$$y = \mu_0 \sin \phi_0 \int_0^x (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{-\frac{1}{2}} dx \ldots\ldots\ldots\ldots (\text{i}).$$

The orthotomic surfaces of all rays from the origin are generated by the revolution round the axis of x of the curves

$$V = \mu_0 y \sin \phi_0 + \int_0^x (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{\frac{1}{2}} dx \ldots\ldots\ldots (\text{ii}),$$

where after integration we must substitute for $\sin \phi_0$ in terms of x and y from the equation of the ray.

If we consider only a small pencil diverging from the origin, the positions at any point of its focal lines will be found as follows.

The orthotomic surface being a surface of revolution, one principal centre of curvature lies at the point where the normal cuts the axis of revolution. The secondary focus therefore lies throughout on the axis of x.

To obtain the distance of the primary focus from the point of incidence of the axis of the pencil, or the radius of curvature of the generating curve, we differentiate equation (i), treating ϕ_0 as a function of x and y; hence

$$\sin \phi_0 (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{-\frac{1}{2}} + \cos \phi_0 \frac{\partial \phi_0}{\partial x} \int_0^x \mu^2 (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{-\frac{3}{2}} dx = 0,$$

$$1 - \cos \phi_0 \frac{\partial \phi_0}{\partial y} \int_0^x \mu^2 (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{-\frac{3}{2}} dx = 0.$$

Since $$\frac{\partial V}{\partial x} = (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{\frac{1}{2}}, \quad \frac{\partial V}{\partial y} = \mu_0 \sin \phi_0,$$

we deduce $$\frac{\partial^2 V}{\partial x^2} = (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{-\frac{1}{2}} \left(\mu \frac{d\mu}{dx} - \mu_0^2 \sin \phi_0 \cos \phi_0 \frac{\partial \phi_0}{\partial x}\right),$$

$$\frac{\partial^2 V}{\partial x \partial y} = \mu_0 \cos \phi_0 \frac{\partial \phi_0}{\partial x}, \quad \frac{\partial^2 V}{\partial y^2} = \mu_0 \cos \phi_0 \frac{\partial \phi_0}{\partial y}.$$

Substitute these values in the formula

$$\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right\}^{\frac{3}{2}} \Big/ \rho = \frac{\partial^2 V}{\partial x^2}\left(\frac{\partial V}{\partial y}\right)^2 - 2\frac{\partial^2 V}{\partial x \partial y}\frac{\partial V}{\partial x}\frac{\partial V}{\partial y} + \frac{\partial^2 V}{\partial y^2}\left(\frac{\partial V}{\partial x}\right)^2,$$

and we obtain

$$\frac{\mu^3}{\rho} = \frac{\mu_0^2 \sin^2 \phi_0}{(\mu^2 - \mu_0^2 \sin^2 \phi)^{\frac{1}{2}}} \mu \frac{d\mu}{dx} + \frac{\mu^4}{\mu^2 - \mu_0^2 \sin^2 \phi} \left[\int_0^x \frac{\mu^2 \, dx}{(\mu^2 - \mu_0^2 \sin^2 \phi_0)^{\frac{3}{2}}}\right]^{-1},$$

or in terms of μ and the angle of incidence ϕ

$$\frac{1}{\rho} = \frac{\sin^2 \phi}{\mu \cos \phi} \frac{d\mu}{dx} + \frac{1}{\mu \cos^2 \phi} \left[\int_0^x \frac{dx}{\mu \cos^3 \phi}\right]^{-1}.$$

187. Maxwell's Theorems.

The chief applications of the characteristic function are due to Clerk Maxwell*, who was the first to discuss in this manner the properties of a symmetrical optical instrument for small pencils and also for pencils with a finite angle of divergence, and the path of a small pencil of rays through an asymmetric instrument and through heterogeneous media. But his method is so different from that actually used by Hamilton, that the following sketch may be useful.

A form of the reduced path, denoted below by U_1 and determined as follows, takes the place of the characteristic function defined in Art. 182.

Let a ray from a point P in a medium of index μ be reflected or refracted at any surfaces and pass through a point P' in a medium of index μ'. Let (x_1, y_1, z_1) be the coordinates of P, (x_1', y_1', z_1') those of P', and let (ξ_1, η_1, ζ_1), $(\xi_2, \eta_2, \zeta_2), \ldots (\xi_n, \eta_n, \zeta_n)$ be those of the points of incidence of the ray on the refracting surfaces. The reduced path from P to P' is equal to

$$\mu\{(\xi_1-x_1)^2+(\eta_1-y_1)^2+(\zeta_1-z_1)^2\}^{\frac{1}{2}}+\mu_1\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{\frac{1}{2}}$$
$$+\ldots+\mu'\{(x_1'-\xi_n)^2+(y_1'-\eta_n)^2+(z_1'-\zeta_n)^2\}^{\frac{1}{2}}.$$

Let this form be denoted by U_0; by Fermat's theorem U_0 is stationary in value for all small displacements of the points of incidence on the refracting surfaces. We may suppose that we substitute, to any required degree of accuracy, for the coordinate ζ in terms of the coordinates ξ and η from the equations of the surfaces.

We have then as many pairs of equations of the type $\frac{\partial U_0}{\partial \xi}=0$, $\frac{\partial U_0}{\partial \eta}=0$, as there are refractions.

Solving these equations we can determine the coordinates ξ, η in terms of those of P and P' to any required degree of accuracy, and obtain on substitution the value of the reduced path in terms of those coordinates only; denote the form so obtained by U_1.

Then the direction-cosines of the ray at P' are given by the equations

$$\frac{\partial U_1}{\partial x_1'}=\mu' l', \quad \frac{\partial U_1}{\partial y_1'}=\mu' m', \quad \frac{\partial U_1}{\partial z_1'}=\mu' n';$$

and those at P by the equations

$$\frac{\partial U_1}{\partial x_1}=-\mu l, \quad \frac{\partial U_1}{\partial y_1}=-\mu m, \quad \frac{\partial U_1}{\partial z_1}=-\mu n.$$

For U_1 differs from U_0 only by the substitution of (ξ, η) in terms of the coordinates of P and P'.

Hence
$$\frac{\partial U_1}{\partial x_1'}=\frac{\partial U_0}{\partial x_1'}+\Sigma\left(\frac{\partial U_0}{\partial \xi}\frac{\partial \xi}{\partial x_1'}+\frac{\partial U_0}{\partial \eta}\frac{\partial \eta}{\partial x_1'}\right)$$
$$=\frac{\partial U_0}{\partial x_1'}=\frac{\mu'(x_1'-\xi_n)}{r'}=\mu' l',$$

* Maxwell, *Collected Papers*, 1890, Vol. II. p. 381, p. 439.

and
$$\frac{\partial U_1}{\partial x_1}=\frac{\partial U_0}{\partial x_1}+\Sigma\left(\frac{\partial U_0}{\partial \xi}\frac{\partial \xi}{\partial x_1}+\frac{\partial U_0}{\partial \eta}\frac{\partial \eta}{\partial x_1}\right)$$
$$=\frac{\partial U_0}{\partial x_1}=-\mu\frac{\xi_1-x_1}{r}=-\mu l.$$

It is clear that this form U_1 of the reduced distance is really equal in value to $V'-V$, but that the transformation from one form to the other is attained by expressing that the points P, P' lie on the same ray. This may be done by solving the six equations of the type
$$\frac{\partial U_1}{\partial x_1'}=\mu' l'=\frac{\partial V'}{\partial x_1'};\ \frac{\partial U_1}{\partial x_1}=-\mu l=-\frac{\partial V}{\partial x_1},$$
and the coefficients involved in V and V' must be such that these six equations give consistent relations between (x_1, y_1, z_1) and (x_1', y_1', z_1').

When the point (x_1, y_1, z_1) is the origin of a pencil of rays, U_1 is actually the characteristic function for the point (x_1', y_1', z_1'); and (x_1, y_1, z_1) are the constants involved in the equation of the original orthotomic surface.

Further, if Q be a point (x, y, z) on the ray in the medium of index μ, Q' a point (x', y', z') on the ray in the medium of index μ', the reduced path from Q to Q', denoted by U, is equal to
$$\mu QP+U_1+\mu' P'Q'$$
$$=\mu\{(x_1-x)^2+(y_1-y)^2+(z_1-z)^2\}^{\frac{1}{2}}+U_1+\mu'\{(x'-x_1')^2+(y'-y_1')^2+(z'-z_1')^2\}^{\frac{1}{2}},$$
and therefore
$$\frac{\partial U}{\partial x_1}=\mu(x_1-x)/QP+\frac{\partial U_1}{\partial x_1}=\mu l-\mu l=0,$$
$$\frac{\partial U}{\partial x_1'}=-\mu'(x'-x_1')/P'Q'+\frac{\partial U_1}{\partial x_1'}=-\mu' l'+\mu' l'=0.$$

Hence U is stationary in value for any arbitrary variations of the points P and P'. This theorem may be regarded as included in Fermat's theorem, the ray being incident at P on any arbitrary surface dividing the medium μ into two parts, and similarly at P'.

188. Symmetrical Optical Instrument.

Take any origins O, O' on the axis of the instrument in the first and last media respectively. Let the axis of the instrument be the axis of z, the positive direction for Oz being opposite to that of the incident light, the positive direction for $O'z'$ being that of the emergent light. (It is convenient to take these axes in opposite directions in order that the analytical expressions for the real distances r, r' may be obviously positive.) Also let the axes Ox, $O'x'$ be parallel, and the axes Oy, $O'y'$.

The reduced path U_1 from a point $(x_1, y_1, 0)$ to a point $(x_1', y_1', 0)$ may be expanded in powers of those coordinates. The constant

term is the reduced path U_{00} from O to O'; the terms of the first degree cannot occur, since the differential coefficients $\dfrac{\partial U_1}{\partial x_1}$ &c. vanish with the coordinates, OO' being the path of a ray; the terms of the second degree must occur in the forms

$$x_1^2 + y_1^2,\quad x_1'^2 + y_1'^2,\ \text{and}\ x_1x_1' + y_1y_1',$$

since, the instrument being symmetrical about its axis, U_1 must be unaltered by any rotation of the axes of reference round the axis.

We therefore have, *to the second order of small quantities*,

$$U_1 = U_{00} + \tfrac{1}{2}\{c_1(x_1^2 + y_1^2) + 2d_1(x_1x_1' + y_1y_1') + c_1'(x_1'^2 + y_1'^2)\} \ldots \text{(i)},$$

where the coefficients c_1, c_1', d_1 are functions of the powers of the refracting surfaces, their distances apart, and the positions of O and O' with regard to them.

Again, let (x, y, z) be any point Q in the first medium, (x', y', z') any point Q' in the last medium. If a ray from Q meet the plane $z=0$ in the point $(x_1, y_1, 0)$, and on emergence cross the plane $z'=0$ at $(x_1', y_1', 0)$, and then pass through Q', the reduced path U from Q to Q' is given by the equation

$$\begin{aligned} U &= \mu\{z^2 + (x_1 - x)^2 + (y_1 - y)^2\}^{\frac{1}{2}} + U_1 \\ &\qquad + \mu'\{z'^2 + (x' - x_1')^2 + (y' - y_1')^2\}^{\frac{1}{2}} \\ &= \mu\left\{z + \frac{(x_1 - x)^2 + (y_1 - y)^2}{2z}\right\} + U_1 \\ &\qquad + \mu'\left\{z' + \frac{(x' - x_1')^2 + (y' - y_1')^2}{2z'}\right\} \ldots\ldots \text{(ii)} \end{aligned}$$

to the second order.

But since U is stationary in value for variations of x_1, y_1, x_1', y_1',

$$\left.\begin{aligned} c_1x_1 + d_1x_1' &= -\mu\,(x_1 - x)/z \\ d_1x_1 + c_1'x_1' &= \mu'\,(x' - x_1')/z' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

with similar equations in y, as throughout this and the following articles.

Solving these equations, we obtain

$$\left.\begin{aligned} D\,x &= (c_1' + \mu'/z')\,\mu x/z - d_1\mu' x'/z' \\ D\,x' &= -d_1\mu x/z + (c_1 + \mu/z)\,\mu' x'/z' \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(iv)},$$

$$\left.\begin{aligned} D\,(x - x_1) &= \{c_1(c_1' + \mu'/z') - d_1^2\}\,x + d_1\mu' x'/z' \\ D\,(x' - x_1') &= d_1\mu x/z + \{c_1'(c_1 + \mu/z) - d_1^2\}\,x' \end{aligned}\right\} \ldots\ldots \text{(v)},$$

where $$D = \{(c_1 + \mu/z)(c_1' + \mu'/z') - d_1^2\}.$$

Again, equation (ii) may be written as

$$U = U_{\infty} + \mu z + \mu' z'$$
$$+ \tfrac{1}{2}\left[x_1 \left\{\frac{\mu}{z}(x_1 - x) + c_1 x_1 + d_1 x_1'\right\} + x_1' \left\{\frac{\mu'}{z'}(x_1' - x') + d_1 x_1 + c_1' x_1'\right\}\right.$$
$$\left. + \frac{\mu}{z} x (x - x_1) + \frac{\mu'}{z'} x' (x' - x_1') + \text{terms in } y\right]$$
$$= U_{\infty} + \mu z + \mu' z' + \tfrac{1}{2}\left\{\frac{\mu}{z} x (x - x_1) + \frac{\mu'}{z'} x' (x' - x_1')\right\} + \tfrac{1}{2}\{(y)\} \text{ by (iii)}$$
$$= U_{\infty} + \mu z + \mu' z' + \tfrac{1}{2}\{c(x^2 + y^2) + 2d(xx' + yy') + c'(x'^2 + y'^2)\} \quad \text{......(vi)},$$

where
$$\left.\begin{aligned} c &= \frac{\mu}{z}\left\{c_1\left(c_1' + \frac{\mu'}{z'}\right) - d_1^2\right\} \Big/ D \\ c' &= \frac{\mu'}{z'}\left\{c_1'\left(c_1 + \frac{\mu}{z}\right) - d_1^2\right\} \Big/ D \\ d &= \frac{\mu\mu'}{zz'} d_1 \Big/ D \end{aligned}\right\} \text{............ (vii)}.$$

Hence if we move the origins distances z and z' outwards along the axis, the coefficients in the expression for the reduced path are changed in the manner given by equations (vii).

It is obvious that U takes its simplest form when z and z' are so chosen that c and c' are zero. If this be done, we find that $d = (d_1^2 - c_1 c_1')/d_1$. This expression is therefore an invariant of the system, and it is easy to verify from (vii) that

$$(d^2 - cc')/d \equiv (d_1^2 - c_1 c_1')/d_1.$$

We denote this invariant by K, and hence, *if K be not zero*, it is possible by moving the first origin the distance $\mu c_1'/d_1 K$ and the second origin the distance $\mu' c_1/d_1 K$ from the instrument to destroy the coefficients of the squares in U. These new origins are, as we shall see, the principal foci of the system; and the reduced path from a point $(x_1, y_1, 0)$ on the first focal plane to a point $(x_1', y_1', 0)$ on the second focal plane is

$$U_{\infty} + K(x_1 x_1' + y_1 y_1') \text{..................... (viii)}.$$

The reduced path U from a point (x, y, z) to a point (x', y', z'), the principal foci being the origins, is easily seen, by making the substitutions $c_1 = c_1' = 0$, $d_1 = K$ in (vii), to be

$$U_{\infty} + \mu z + \mu' z' + \tfrac{1}{2}\frac{\mu z'(x^2 + y^2) - 2\mu f'(xx' + yy') + \mu' z(x'^2 + y'^2)}{zz' - ff'} \quad \text{......(ix)},$$

where we have written f for μ/K and f' for μ'/K.

Also equations (iii) now take the form

$$\left.\begin{aligned} Kx_1' &= -\mu\,(x_1 - x)/z = -\mu l \\ Kx_1 &= \mu'\,(x' - x_1')/z' = \mu' l' \end{aligned}\right\} \quad\text{(x)},$$

whence we deduce

$$x = -lz + f'l', \quad x' = l'z' - fl \quad\text{(xi)},$$

and the initial and final direction-cosines of the ray from (x, y, z) to (x', y', z') are given by the equations

$$\left.\begin{aligned} l &= (f'x' - z'x)/(zz' - ff') \\ l' &= (zx' - fx)/(zz' - ff') \end{aligned}\right\} \quad\text{(xii)}$$

with similar equations in y and in m.

189. Cardinal Points.

Principal foci. If in equations (x) $l = m = 0$, it follows that $x_1' = y_1' = 0$, *i.e.* any ray incident parallel to the axis passes through the second origin; this point is therefore the second principal focus. Similarly the first origin is the first principal focus.

If l and m be given, x_1', y_1' are known; any incident pencil of parallel rays therefore converges to a focus on the second focal plane.

Unit points and planes. If $z = f$, $z' = f'$, it follows from (xi) that $x = x'$ and $y = y'$ independently of the values of l and l', m and m'. Hence every ray from a point on the first unit plane given by $z = f$ meets the second unit plane $z' = f'$ in a point in the same axial plane at an equal distance from the axis. For points on these planes equations (x) reduce to one,

$$x = x' = x_1 + x_1',$$

and equations (xi) reduce to $\mu' l' - \mu l = Kx$. (Cf. Art. 80.) The focal lengths f and f' are the distances between the unit planes and the focal planes; hence $f = \mu/K$, $f' = \mu'/K$.

Nodal points. If $z = f'$ and $z' = f$, x and x' vanish together for any equal values of l and l'. Hence any incident ray crossing the axis at the first nodal point crosses the axis on emergence at the second nodal point in the same direction.

Apparent distance. If $x' = 0$, equation (xii) gives

$$\frac{x}{-l'} = \frac{zz' - ff'}{f}.$$

This is an expression for the distance from the eye at which an object of height x must be placed to subtend the same angle, when seen directly, that it appears to subtend through the instrument, the distance of the object in front of F_1 being z, and of the eye behind F_2 being z'; it is therefore the apparent distance.

190. Conjugate Foci.

If $zz' = ff'$, equations (x), which may be written

$$\frac{x_1}{z} + \frac{x_1'}{f} = \frac{x}{z} \text{ and } \frac{x_1}{f'} + \frac{x_1'}{z'} = \frac{x'}{z'},$$

are inconsistent unless in addition

$$\frac{x'}{x} = \frac{f}{z} = \frac{z'}{f'} = \frac{y'}{y}.$$

If these relations do not hold, no ray that leaves the point (x, y, z) can, to this order of approximation, pass through the point (x', y', z') on the conjugate plane; but if these relations hold, equations (x) are equivalent to one only, and all rays from the origin of light pass through its geometrical focus. The conjugate foci lie in the same axial plane, since $x'/y' = x/y$; and the linear magnification is f/z or z'/f'. The direction-cosines of the ray as given by equations (xii) are indeterminate; but the initial and final direction-cosines of any ray through the conjugate foci are connected by the single equation (xi)

$$f'l' = x + lz = x_1.$$

The form of the reduced path given in (ix) appears to become infinite, but in reality equations (iii) or (x) from which it was deduced are here equivalent to one only, and it will be found on repeating the work by which U was calculated that the reduced path between two conjugate foci is

$$U_{00} + \mu z + \mu' z' + \tfrac{1}{2} K (xx' + yy').$$

Since x and x' are not independent, this form may not be differentiated.

191. Normal adjustment.

When the invariant K of the system is zero, or $d^2 = cc'$, we find on making this substitution in (vii) that if the reduced path between the two points $(x_1, y_1, 0)$, $(x_1', y_1', 0)$ be

$$U_{00} + \tfrac{1}{2} \{c_1 (x_1^2 + y_1^2) + 2d_1 (x_1 x_1' + y_1 y_1') + c_1' (x_1'^2 + y_1'^2)\} \ldots\ldots\text{(i)},$$

the reduced path between the two points (x, y, z), (x', y', z') is

$$U_{00} + \mu z + \mu' z' + \tfrac{1}{2} \{c (x^2 + y^2) + 2d (xx' + yy') + c' (x'^2 + y'^2)\} \ldots\ldots\ldots\text{(ii)},$$

where $$c/c_1 = d/d_1 = c'/c_1' = 1/(1 + c_1 z/\mu + c_1' z'/\mu') \ldots\ldots\ldots\text{(iii)}.$$

Hence no change of origin can alter the common value of the ratios c'/d and d/c; this is therefore a constant of the instrument, which we shall denote by $-M$.

In this case equations (iii) or (x) take the form

$$\left.\begin{aligned} -\mu l &= -\mu\ (x_1 - x)/z = d_1(x_1' - x_1/M) = d\,(x' - x/M) \\ \mu' l' &= \quad \mu'\ (x' - x_1')/z' = d_1(x_1 - Mx_1') = d\,(x - Mx') \end{aligned}\right\} \text{......(iii}'\text{)}.$$

Hence $\mu' l'/\mu l = M = \mu' m'/\mu m$; *i.e.* the angle any ray makes with the axis is changed in a constant ratio.

The relation between conjugate foci is given as before by the fact that if $1 + c_1 z/\mu + c_1' z'/\mu' = 0$, equations (iii′) are inconsistent unless $x'/x = y'/y = 1/M$, and that then they are equivalent to one equation only. Hence conjugate foci lie in the same axial plane, and the linear magnification is constant and equal to $1/M$. The reduced path between two conjugate foci

$$\begin{aligned} &= U_{00} + \mu z + \mu' z' + \tfrac{1}{2}\left\{\frac{\mu x}{z}(x - x_1) + \frac{\mu' x'}{z'}(x' - x_1')\right\} \\ &= U_{00} + \mu z + \mu' z' + \tfrac{1}{2}(\mu' l' x' - \mu l x) \\ &= U_{00} + \mu z + \mu' z'. \end{aligned}$$

192. *Given the refracting surfaces, to determine the constants of the reduced path for a symmetrical optical instrument.*

Let there be n coaxial spherical surfaces separating media of indices $\mu, \mu_1, \mu_2 \dots \mu_{n-1}, \mu'$; let their distances apart be $t_1, t_2 \dots t_{n-1}$, and let their radii of curvature in succession be $\rho_1, \rho_2 \dots \rho_n$, so that the equation of any surface referred to its vertex as origin is approximately $2\rho\zeta = \xi^2 + \eta^2$, ρ being positive if the surface be concave to the light.

Take the first vertex as the origin O, the last vertex as the origin O', the axes Oz and $O'z'$ being directed away from the instrument. If U be the reduced path from any point (x, y, z) in the first medium to a point (x', y', z') in the last,

$$\begin{aligned} U &= \mu\{(\xi_1 - x)^2 + (\eta_1 - y)^2 + (\zeta_1 - z)^2\}^{\frac{1}{2}} + \mu_1\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (t_1 + \zeta_1 - \zeta_2)^2\}^{\frac{1}{2}} \\ &\quad + \dots + \mu'\{(x' - \xi_n)^2 + (y' - \eta_n)^2 + (z' + \zeta_n)^2\}^{\frac{1}{2}} \\ &= \mu\left\{z + \frac{x^2 + y^2}{2z} - \frac{x\xi_1 + y\eta_1}{z} + \tfrac{1}{2}(\xi_1^2 + \eta_1^2)\left(\frac{1}{z} - \frac{1}{\rho_1}\right)\right\} \\ &\quad + \mu_1\left\{t_1 + \tfrac{1}{2}(\xi_1^2 + \eta_1^2)\left(\frac{1}{t_1} - \frac{1}{\rho_1}\right) + \tfrac{1}{2}(\xi_2^2 + \eta_2^2)\left(\frac{1}{t_1} - \frac{1}{\rho_2}\right) - \frac{\xi_1\xi_2 + \eta_1\eta_2}{t_1}\right\} \\ &\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\quad + \mu'\left\{z' + \frac{x'^2 + y'^2}{2z'} - \frac{x'\xi_n + y'\eta_n}{z'} + \tfrac{1}{2}(\xi_n^2 + \eta_n^2)\left(\frac{1}{z'} + \frac{1}{\rho_n}\right)\right\} \dots\dots\dots\dots\text{(i)} \end{aligned}$$

to the second order of small quantities.

By Fermat's theorem, $\dfrac{\partial U}{\partial \xi}=\dfrac{\partial U}{\partial \eta}=0$ for all suffixes.

If U_2 denote the terms in U of the second order in ξ and η, we have

$$\frac{\partial U_2}{\partial \xi_1}=\frac{\mu x}{z},\quad \frac{\partial U_2}{\partial \xi_n}=\frac{\mu' x'}{z'},\ \text{and}\ \frac{\partial U_2}{\partial \xi_r}=0,$$

for all values of the suffix r from 2 to $(n-1)$ inclusive.

Again $$2U_2\equiv\left(\xi_1\frac{\partial U_2}{\partial \xi_1}+\ldots+\xi_n\frac{\partial U_2}{\partial \xi_n}\right)+\left(\eta_1\frac{\partial U_2}{\partial \eta_1}+\ldots+\eta_n\frac{\partial U_2}{\partial \eta_n}\right)$$
$$=\mu\,\frac{x\xi_1+y\eta_1}{z}+\mu'\,\frac{x'\xi_n+y'\eta_n}{z'}.$$

Hence

$$U=\mu z+\Sigma\mu t+\mu' z'+\mu\,\frac{x(x-\xi_1)+y(y-\eta_1)}{2z}+\mu'\,\frac{x'(x'-\xi_n)+y'(y'-\eta_n)}{2z'}\ldots\text{(ii)}.$$

It is therefore only necessary to determine the coordinates of the first and last points of incidence of the ray.

The equations of the type $\dfrac{\partial U}{\partial \xi}=0$ are

$$\left(\frac{\mu_1-\mu}{\rho_1}+\frac{\mu_1}{t_1}+\frac{\mu}{z}\right)\xi_1-\frac{\mu_1}{t_1}\xi_2 \qquad\qquad =\frac{\mu x}{z}$$

$$-\frac{\mu_1}{t_1}\xi_1+\left(\frac{\mu_2-\mu_1}{\rho_2}+\frac{\mu_1}{t_1}+\frac{\mu_2}{t_2}\right)\xi_2-\frac{\mu_2}{t_2}\xi_3 \qquad\qquad =0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\frac{\mu_{n-1}}{t_{n-1}}\xi_{n-1}+\left(\frac{\mu'-\mu_{n-1}}{\rho_n}+\frac{\mu_{n-1}}{t_{n-1}}+\frac{\mu'}{z'}\right)\xi_n=\frac{\mu' x'}{z'}.$$

Let D denote the symmetrical determinant

$$\begin{vmatrix} \kappa_1+\mu_1/t_1, & -\mu_1/t_1, & 0, & 0 \\ -\mu_1/t_1, & \kappa_2+\mu_1/t_1+\mu_2/t_2, & -\mu_2/t_2, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & -\mu_{n-1}/t_{n-1}, & \kappa_n+\mu_{n-1}/t_{n-1} \end{vmatrix}.$$

Also let D_1, D_1' denote the co-factors of the first and last elements of the leading diagonal, and T the co-factor of the last element of the first row or of the first column. Let D_2 denote the determinant obtained by removing the first and last rows and columns.

We have $\begin{vmatrix} D_1, & T \\ T, & D_1' \end{vmatrix}\equiv DD_2$; also $T=\dfrac{\mu_1\mu_2\ldots\mu_{n-1}}{t_1t_2\ldots t_{n-1}}$.

On solving the equations above we obtain

$$\Delta\xi_1=\frac{\mu x}{z}\left(D_1+\frac{\mu'}{z'}D_2\right)+\frac{\mu' x'}{z'}T,$$

$$\Delta\xi_n=\frac{\mu x}{z}T+\frac{\mu' x'}{z'}\left(D_1'+\frac{\mu}{z}D_2\right),$$

where Δ is the determinant formed of the complete coefficients in the equations, *i.e.* by Taylor's theorem

$$\Delta = D + \frac{\mu}{z}\frac{\partial D}{\partial \kappa_1} + \frac{\mu'}{z'}\frac{\partial D}{\partial \kappa_n} + \frac{\mu\mu'}{zz'}\frac{\partial^2 D}{\partial \kappa_1 \partial \kappa_n},$$
$$= D + \frac{\mu}{z} D_1 + \frac{\mu'}{z'} D_1' + \frac{\mu\mu'}{zz'} D_2.$$

Hence

$$\mu\frac{x-\xi_1}{z} = \left\{x\left(\frac{z'}{\mu'}D + D_1'\right) - x'T\right\} \Big/ \left\{\frac{zz'}{\mu\mu'}D + \frac{z'}{\mu'}D_1 + \frac{z}{\mu}D_1' + D_2\right\},$$

and a similar expression holds for $\mu'(x'-\xi_n)/z'$.

On substituting in U we obtain, making use in the denominator of the identity above,

$$U = U_{00} + \mu z + \mu' z' + \frac{\left(\frac{z'}{\mu'}D + D_1'\right)(x^2+y^2) - 2T(xx'+yy') + \left(\frac{z}{\mu}D + D_1\right)(x'^2+y'^2)}{2\left\{\left(\frac{z}{\mu}D + D_1\right)\left(\frac{z'}{\mu'}D + D_1'\right) - T^2\right\} \Big/ D}.$$

This value agrees with the form obtained in Art. 188 for the reduced path between any two points, when that between two points on the planes of reference is given.

The positions of the principal foci are given by the equations $z/\mu = -D_1/D$, $z'/\mu' = -D_1'/D$; and on moving the origins to these points, the coefficient of the term $(xx'+yy')$, which is the power K, is equal to D/T or $\frac{t_1 t_2 \ldots t_{n-1}}{\mu_1\mu_2\ldots\mu_{n-1}} D$.

In calculating D, D_1 or $\frac{\partial D}{\partial \kappa_1}$, D_1' or $\frac{\partial D}{\partial \kappa_n}$, the minus signs may be removed from all the terms adjacent to the leading diagonal.

193. *A small pencil of light diverging from two focal lines near the axis of the instrument passes through a symmetrical optical instrument; to determine the focal lines of the emergent pencil.*

Let the principal foci of the instrument be the origins in each medium; and let the equation of the orthotomic surface through F_1 of the incident rays be

$$z = \alpha x + \beta y + \tfrac{1}{2}(ax^2 + 2hxy + by^2),$$

where α and β are small quantities of the first order. (This is the general form, for there is no reason here to suppose that the axis of the instrument is one of the rays of this pencil; and it is necessary to point out that, unless the equations of the orthotomic surfaces and the expression of the reduced path through the instrument be given to powers of the coordinates higher than the second, it is impossible to proceed any further than the first approximation of focal lines, or to say that one ray more than another is the axis of the pencil.)

Similarly let the equation of the orthotomic surface through F_2 of the emergent rays be

$$z' = \alpha' x' + \beta' y' + \tfrac{1}{2}(a' x'^2 + 2h' x' y' + b' y'^2).$$

The reduced path between two points $(x_1, y_1, 0)$, $(x_1', y_1', 0)$ on the focal planes is $U_\infty + K(x_1 x_1' + y_1 y_1')$.

Hence if a ray normal to the first surface at (x, y) cross the first focal plane in (x_1, y_1), and, crossing the second focal plane at (x_1', y_1'), be normal to the second surface at (x', y'), we have

$$\left.\begin{aligned} Kx_1' &= -\ \mu l &= -\mu\,(\alpha + ax + hy) \\ Ky_1' &= -\ \mu m &= -\mu\,(\beta + hx + by) \end{aligned}\right\},$$

$$\left.\begin{aligned} Kx_1 &= \ \ \mu' l' &= -\mu'(\alpha' + a'x' + h'y') \\ Ky_1 &= \ \ \mu' m' &= -\mu'(\beta' + h'x' + b'y') \end{aligned}\right\},$$

the negative sign occurring in the second pair of equations, because the third direction-cosine of the emergent ray is $+1$.

But the difference of the values of x and x_1, y and y_1 is only of the third order of small quantities; hence the equations above, which are only approximations, may be written

$$\left.\begin{aligned} \alpha + ax + hy + x'/f &= 0 \\ \beta + hx + by + y'/f &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \alpha' + a'x' + h'y' + x/f' &= 0 \\ \beta' + h'x' + b'y' + y/f' &= 0 \end{aligned}\right\},$$

and these equations must be consistent. Solving therefore for x and y from the first pair and substituting in the second pair, we obtain identities, whence the constants of the emergent rays are given by the equations

$$\frac{a'}{b} = \frac{h'}{-h} = \frac{b'}{a} = \frac{1/ff'}{ab - h^2},$$

$$\alpha' = (b\alpha - h\beta)/(ab - h^2)f', \quad \beta' = (a\beta - h\alpha)/(ab - h^2)f'.$$

If the equation of the orthotomic surface referred to an origin other than F_1 be required, it is easy to prove, by determining the ray that cuts the axis at $z = u$, and making use of Art. 171, that the equation of the orthotomic surface through that point is

$$\Delta(z - u) = \{\alpha - u(b\alpha - h\beta)\}\,x + \{\beta - u(a\beta - h\alpha)\}\,y$$
$$+ \tfrac{1}{2}\{ax^2 + 2hxy + by^2 - (ab - h^2)\,u\,(x^2 + y^2)\},$$

where $\Delta = 1 - (a + b)\,u + (ab - h^2)\,u^2 = (1 - u/v_1)(1 - u/v_2)$.

194. The properties of a small pencil of rays passing through an asymmetric optical system, consisting either of any surfaces having a common axis, or more generally of a heterogeneous medium bounded by initial and final homogeneous media, are investigated by the use of the characteristic function in papers by Clerk Maxwell (*Collected Papers*, Vol. II. p. 381), by Mr J. Larmor (*Proc. London Mathematical Society*, Vol. XX. 1889, p. 181, Vol. XXIII. 1892, p. 165), and by the author (*Quarterly Journal*, Vol. XXVII. 1895, p. 191). They are also the subject of a paper by Mr Sampson (*Proc. London Mathematical Society*, Vol. XXIX. 1898, p. 33), following the method of Gauss' *Dioptrische Untersuchungen.*

The second approximation to the reduced path in a symmetrical optical instrument, and its application to Aberration and Distortion will be found below, Chap. XIV.

EXAMPLES.

1. The equation of the orthotomic surface of a small pencil at the origin is

$$z=\tfrac{1}{2}(ax^2+2hxy+by^2)+\tfrac{1}{3}(cx^3+3dx^2y+3exy^2+fy^3).$$

Prove that, if such a pencil be refracted at a plane surface, the following quantities are unaltered by refraction;

$$\mu\sin\phi,\ \mu a\cos^2\phi,\ \mu h\cos\phi,\ \mu b,\ \mu\{\tfrac{1}{2}(a^2+h^2)\sin\phi+\tfrac{1}{3}c\cos\phi\}\cos^2\phi,$$
$$\mu\{(a+b)h\sin\phi\cos\phi+d\cos^2\phi\},\ \mu\{(b^2+h^2)\sin\phi+e\cos\phi\},\ \mu f.$$

2. The equations of the surfaces of a lens of thickness τ and index μ are

$$2z=rx^2+2sxy+ty^2,\ \text{and}\ 2z'=r'x'^2+2s'x'y'+t'y'^2,$$

the origins being at the points where the axis cuts the lens, and the axes of z and z' being drawn away from the lens. Shew that the reduced path from a point $(x, y, 0)$ near the axis to a similar point $(x', y', 0)$ is approximately

$$\mu\tau+\tfrac{1}{2}(\mu-1)(rx^2+2sxy+ty^2+r'x'^2+2s'x'y'+t'y'^2)$$
$$+\tfrac{1}{2}\mu(x^2+y^2+x'^2+y'^2-2xx'-2yy')/\tau.$$

3. The reduced path between two points $(x, y, 0)$, $(x', y', 0)$ near the axis of an asymmetric optical instrument is given approximately as

$$U=U_0+\tfrac{1}{2}\{c_{11}x^2+2c_{12}xy+c_{22}y^2+c'_{11}x'^2+2c'_{12}x'y'+c'_{22}y'^2+2d_{11}xx'+2d_{22}yy'\}.$$

A small pencil of rays orthogonal to the surface $z=\tfrac{1}{2}(ax^2+2hxy+by^2)$ passes through the instrument; shew that the emergent pencil is orthogonal to the

surface $z' = \frac{1}{2}(a'x'^2 + 2h'x'y' + b'y'^2)$, where the coefficients are given by the equations

$$\frac{(\mu b + c_{12})\, d_{11}^2}{\mu' a' + c_{11}'} = \frac{(\mu a + c_{11})\, d_{22}^2}{\mu' b' + c_{22}'} = -\frac{(\mu h + c_{12})\, d_{11} d_{22}}{\mu' h' + c_{12}'}$$
$$= (\mu a + c_{11})(\mu b + c_{22}) - (\mu h + c_{12})^2.$$

4. The reduced path between two points $(x, y, 0)$, $(x', y', 0)$ near the axis of an asymmetric optical instrument being given in the form above, shew that the reduced path between two points (ξ, η, u), (ξ', η', u') is

$$U_0 + \mu u + \mu' u' + \frac{\mu}{2u}(\xi^2 + \eta^2) + \frac{\mu'}{2u'}(\xi'^2 + \eta'^2) + \tfrac{1}{2} U_2,$$

where

$$U_2 \begin{vmatrix} c_{11} + \mu/u & c_{12} & d_{11} & 0 \\ c_{12} & c_{22} + \mu/u & 0 & d_{22} \\ d_{11} & 0 & c_{11}' + \mu'/u' & c_{12}' \\ 0 & d_{22} & c_{12}' & c_{22}' + \mu'/u' \end{vmatrix}$$
$$= \begin{vmatrix} c_{11} + \mu/u & c_{12} & d_{11} & 0 & \mu\xi/u \\ c_{12} & c_{22} + \mu/u & 0 & d_{22} & \mu\eta/u \\ d_{11} & 0 & c_{11}' + \mu'/u' & c_{12}' & \mu'\xi'/u' \\ 0 & d_{22} & c_{12}' & c_{22}' + \mu'/u' & \mu'\eta'/u' \\ \mu\xi/u & \mu\eta/u & \mu'\xi'/u' & \mu'\eta'/u' & 0 \end{vmatrix}.$$

5. The surfaces of equal density of a heterogeneous medium all meet the axis of z at right angles, and a small pencil of rays, passing through two focal lines and having this line as axis, passes through the medium. Prove that at any point where the principal radii of curvature of the refracting surface are ρ_1 and ρ_2, and the distances of the focal lines from this point are v_1 and v_2, the principal planes of the pencil are turning round at a rate given by

$$\frac{d\theta}{dz} = \frac{1/\rho_1 - 1/\rho_2}{1/v_1 - 1/v_2} \frac{\sin 2\omega}{\mu} \frac{d\mu}{dz},$$

where ω is the angle between a principal plane of curvature and a principal plane of the pencil.

6. If u be the distance of an origin of light from the point of incidence of a ray on a refracting sphere, ϕ and ϕ' the angles of incidence and refraction of that ray, prove that a ray incident in the same plane at an adjacent point of the sphere at angular distance θ will after refraction intersect the refracted ray at distance v from the point of incidence given by

$$\mu \cos\phi \left(\frac{1}{\rho} + \frac{\cos\phi}{u}\right)\left(1 + \frac{3\rho\theta \sin\phi}{u}\right) = \mu' \cos\phi' \left(\frac{1}{\rho} + \frac{\cos\phi'}{v}\right)\left(1 + \frac{3\rho\theta \sin\phi'}{v}\right),$$

neglecting powers of θ above the first.

7. A small pencil of light is refracted at any surface, the plane of incidence of the axis being the primary focal plane of the pencil and also a principal plane of the surface at the point of incidence. R_1, P_1 are the radii of curvature of the section of the surface by the primary plane and of the evolute of this section respectively; R_2, P_2 denote similar quantities in the secondary plane; ρ_1, ρ_2 are the radii of curvature of the sections of the

caustic surface of the incident pencil at the focal lines by the primary and secondary planes respectively; ρ_1', ρ_2' denote similar quantities in the refracted pencil. Shew that with the usual notation

$$\mu' \cos \phi' - \mu \cos \phi = R_1 \left\{ \frac{\mu' \cos^2 \phi'}{v_1'} - \frac{\mu \cos^2 \phi}{v_1} \right\} = R_2 \left(\frac{\mu'}{v_2'} - \frac{\mu}{v_2} \right)$$

$$= \frac{R_1^3}{P_1} \left[\left(\frac{\mu' \rho_1' \cos^3 \phi'}{v_1'^3} - \frac{3\mu' \cos^2 \phi' \sin \phi'}{v_1'^2} + \frac{3\mu \cos \phi' \sin \phi'}{v_1' R_1} \right) - \left(\frac{\mu \rho_1 \cos^3 \phi}{v_1^3} - \frac{3\mu \cos^2 \phi \sin \phi}{v_1^2} + \frac{3\mu \cos \phi \sin \phi}{v_1 R_1} \right) \right]$$

$$= \frac{R_2^3}{P_2} \left[\frac{\mu' \rho_2'}{v_2'^3} - \frac{\mu \rho_2}{v_2^3} \right].$$

8. If a system of coaxial refracting surfaces have their principal planes of curvature all coincident with the principal planes of a pencil incident directly, and if S_R be the area of the cross-section of the pencil at any point R in a medium of index μ, Q_1, Q_2 the primary and secondary foci of the pencil in that medium, $\mu^2 S_R l_1 l_2 / Q_1 R \,.\, Q_2 R$ is a constant; where l_1 is the length of the image, formed by primary foci in that medium, of a small line in the primary plane at the primary focus, and l_2 that of the image, formed by secondary foci in that medium, of a small line in the secondary plane at the secondary focus.

9. A congruence of rays is reflected at any surface; prove that the necessary and sufficient condition that the incident rays which form a developable surface may be reflected to form a developable surface is that the tangent and normal planes to the developable through the generator intersect the surface in directions conjugate with regard to the indicatrix at the point of incidence. (Darboux.)

10. Shew that if the normals to a surface all pass through a given curve, one system of lines of curvature are circles, and those normals which pass through a given point are generators of a right cone whose axis is the tangent at that point. Hence shew that, if the normals all pass through two curves, these curves must be conics in planes at right angles to each other, the vertices of either being the foci of the other; and the surface will be a cyclide. (Maxwell.)

CHAPTER XII.

CAUSTICS.

195. A SYSTEM of rays diverging from a point, or orthogonal to a surface, will be orthogonal to a surface after any number of reflections or refractions. Adjacent rays will intersect only if drawn through points lying on a line of curvature of an orthotomic surface; and the rays drawn through any line of curvature will envelope a caustic curve lying on the surface of centres of the orthotomic surface. There will be two foci on each ray, the points of contact of that ray with the surface of centres; and the rays may be grouped in two ways to form developable surfaces, having as their edges of regression these caustic curves on the two sheets of the surface of centres.

The image of an origin of light, which is seen by the eye, will be determined by the small pencil that enters the pupil; the ray to the nodal centre of the eye will touch the surface of centres in two points, and the other rays of the pencil will pass through very small curves on the surface of centres at these points, which are perpendicular to the axial ray and are the focal lines of the small pencil.

The properties of a pencil of rays after reflection at any surface are fully discussed in Darboux, *Théorie des Surfaces*, 1894, t. II. Chap. XIII. We shall in the present chapter consider chiefly the caustic curves enveloped by rays emanating in one plane from an origin, and reflected or refracted at certain curves in that plane.

196. Aplanatic surfaces.

A surface separating media of refractive indices μ and μ' will be aplanatic, *i.e.* it will refract rays from a point Q accurately to a point Q', if the distances of any point of the surface from Q and Q' be connected by the equation $\mu r + \mu' r' = c$.

The general form therefore is that of the surface of revolution generated by this Cartesian oval; if however one of the foci be virtual, r and r' have opposite signs, and the constant c may be taken zero; thus we see that a spherical surface is aplanatic for two points inverse to each other in the sphere (cf. Art. 41).

If the origin Q be at infinity, or the incident pencil consist of parallel rays, the aplanatic surface is given by the equation $\mu p + \mu' r' = c$, where p is the perpendicular from any point of the surface on a plane perpendicular to the rays.

Taking the constant zero, so that either p or r' is virtual, we see that one aplanatic surface is a prolate quadric of revolution of eccentricity μ/μ', having the focus Q' at one of its foci, and the orthotomic plane as the corresponding directrix plane, so that the rays are parallel to the axis of revolution.

In the case of an ellipse the rays are incident on only the further half of the curve, and similarly in the hyperbola on the further branch; and if as in fig. 89 the rays are incident on the convex side of the ellipse in the direction PM, p is virtual, and they are brought to a real focus at Q'; but if they are incident on the concave side in the direction NP', r' is virtual, and they are refracted to form a pencil apparently diverging from Q'.

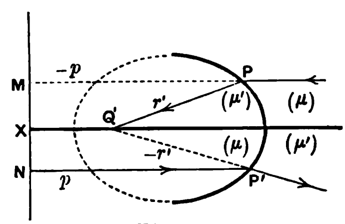


Fig. 89.

197. All caustics are rectifiable.

Every caustic, plane or twisted, is an evolute of the orthotomic curves, and the length of any continuous portion of a caustic is therefore equal to the difference of the intercepts on the extreme rays between an orthotomic curve and their points of contact with the caustic.

These orthotomic curves have been called *secondary caustics* or *anticaustics*. They are at once determined by Malus' theorem. The length of any part of a caustic can be found if we know the characteristic function V for its ends.

Let Q and Q' be two points on the evolute with no cusp or asymptote intervening between them, R and R' the corresponding points on an orthotomic curve in the same medium of index μ.

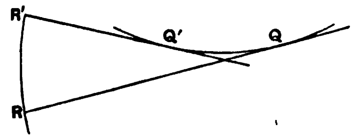


Fig. 90.

We have $V_Q = V_R + \mu RQ,$

$$V_{Q'} = V_{R'} + \mu R'Q'.$$

But $V_R = V_{R'}$, and therefore $V_Q - V_{Q'} = \mu (RQ - R'Q')$.

Hence the arc QQ' of the caustic is equal to $(V_Q - V_{Q'})/\mu$.

Here the characteristic function for Q, or the reduced path for the ray through Q, must be expressed in terms of the coordinates of Q alone, or at least of a parameter defining the position of Q on the caustic. The coordinates of Q may be found if necessary by successive applications of the formulae for primary foci.

198. Application of the reduced path to determine the caustic.

Considering for simplicity the case of a plane caustic, let a ray from O be refracted at n curves in the same plane, and let U be the reduced path $\Sigma(\mu r)$ from O to a point Q on the ray in the final medium. Let U be expressed in terms of parameters $p_1, p_2 \dots p_n$, defining the points of incidence, and of the coordinates of Q.

By Fermat's theorem the n equations

$$\frac{\partial U}{\partial p_1} = \frac{\partial U}{\partial p_2} = \dots = \frac{\partial U}{\partial p_n} = 0 \dots\dots\dots\dots\dots\dots\dots\dots\text{(i)}$$

determine the path of the ray.

But if Q lie on the caustic, the next ray also passes through Q. Hence these equations also hold when the parameters $p_1, p_2 \dots p_n$ are varied, the variations being connected by n equations of the type

$$\frac{\partial^2 U}{\partial p_1{}^2}\partial p_1 + \frac{\partial^2 U}{\partial p_1 \partial p_2}\partial p_2 + \dots + \frac{\partial^2 U}{\partial p_1 \partial p_n}\partial p_n = 0 \dots\dots\dots\text{(ii)}.$$

It follows that

$$\frac{\partial\left(\frac{\partial U}{\partial p_1}, \frac{\partial U}{\partial p_2} \cdots \frac{\partial U}{\partial p_n}\right)}{\partial(p_1, p_2 \cdots p_n)} = 0 \quad \text{(iii)};$$

and if $p_1, p_2 \ldots p_n$ be eliminated between this equation and equations (i), the result is the equation of the caustic.

The fact that the Jacobian vanishes for a point on the caustic shews that the reduced path to such a point is not a true maximum or minimum for arbitrary variations of the points of incidence, but that the value for an arbitrary adjacent path can be made to differ from its value taken from O to Q in excess or defect by small quantities of the third order.

The reduced path from O to an arbitrary point P on the ray will be a minimum or maximum as P precedes or follows the point of contact Q of the ray with the caustic.

Let the ray on which P lies meet an orthotomic curve in R (fig. 90), and let R' be now an adjacent point on the orthotomic curve so that Q is the ultimate point of intersection of the rays through R and R'.

The reduced path for the arbitrary course $O \ldots R'P$ exceeds the reduced path for the actual course $O \ldots RP$ by $\mu(R'P - RP)$ *i.e.* by $\mu(R'P - RQ + PQ)$. But since RQ and $R'Q$ differ only by small quantities of the third order, this is equal to

$$\mu(R'P - R'Q + PQ),$$

and is positive if P precede Q, but if P lie beyond Q, it is equal to $\mu(R'P - R'Q - QP)$, and is negative.

If the path of the ray be not in one plane, there will be two foci Q_1, Q_2 lying on it; and the reduced path from O to a point P may be a maximum for some displacements, and a minimum for others, if P lie between Q_1 and Q_2.

The above is therefore the correct form of Fermat's theorem.

CAUSTIC BY REFLECTION.

199. When a pencil of rays from a point Q is reflected at a given curve, the orthotomic curves are determined by taking R on the reflected ray so that $QP + PR =$ constant, where P is the point of incidence. If the constant be put zero, we see that PR must be taken equal to QP on the reflected ray produced backwards, *i.e.* R is the image of Q in the tangent (fig. 91).

Hence the simplest orthotomic curve is a curve similar to the pedal with regard to Q, but of twice the dimensions; or it may be described as the roulette traced out by a carried point attached to a curve equal to the reflector and rolling on it, the carried point occupying the same position in the rolling curve as the radiant point in the reflector, and the curves always touching at corresponding points.

By the formula for the primary focus after reflection, or by considering Q and Q' as the foci of an ellipse osculating the given curve at P, the distance, v, from P of the point of contact with the caustic is given by the equation

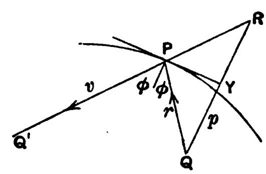


Fig. 91.

$$1/r + 1/v = 2/\rho \cos\phi = 2r/p\rho.$$

Hence $$v = p\rho r/(2r^2 - p\rho),$$

and $$RQ' = 2r^3/(2r^2 - p\rho).$$

The arc of the caustic between any two points is therefore the difference of the values of $[2r^3/(2r^2-p\rho)]$ for the corresponding points of the reflector, no cusp or asymptote of the caustic intervening in its arc.

200. **Caustic by Reflection at a circle.**

The pedal of a circle with regard to any point is a limaçon. It follows from the previous article that the caustic is the evolute of a limaçon of twice the size, or of a certain epitrochoid, in which the fixed and rolling circles are equal. If the radiant point be on the circle, the pedal is a cardioid (a one-cusped epicycloid), and the caustic is therefore also a cardioid. The vertex of the cardioid

is at the radiant point; the cusp of the cardioid is the geometrical focus of rays reflected at the other end of the diameter through the radiant point.

The construction used above fails if the radiant point be at infinity or the incident rays be parallel.

In this case let a ray incident on the reflecting circle at P cut the diameter CB, perpendicular to the rays, in M.

An orthotomic curve is obtained by taking R on the reflected ray so that $PR = PM$. Hence $PR = a \cos \phi$, and therefore PR is a chord of a circle of radius $\frac{1}{2}a$, touching the reflector at P. Also it is easy to see that the arc PR of this circle = the arc PB of the reflector. The orthotomic curve is therefore the two-cusped epicycloid, generated by the rolling of the circle of radius $\frac{1}{2}a$ on the circle of radius a; and the caustic is a similar epicycloid, in which the radius of the fixed circle is $\frac{1}{2}a$ and of the rolling circle $\frac{1}{4}a$.

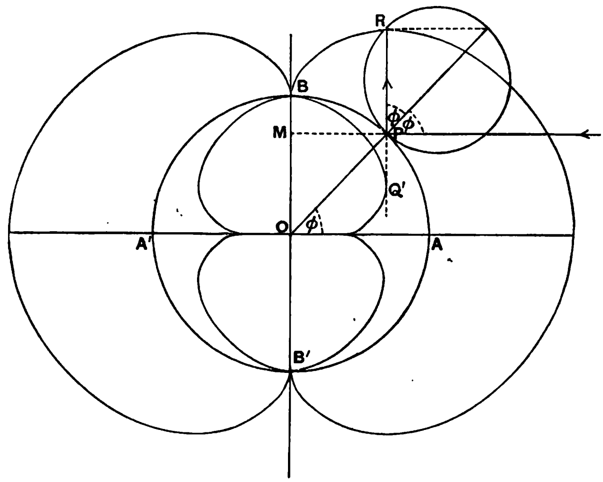


Fig. 92.

In the figure the outer epicycloid is the orthotomic curve, the inner one the caustic. The orthotomic curve is real and the caustic virtual for rays reflected at the convex part of the circle BAB'; the orthotomic curve is virtual and the caustic real for rays reflected at the concave part $BA'B'$. By the formula for primary focus at reflection $PQ' = \frac{1}{2}PR$.

201. The equation of the caustic by reflection at a circle in the general case has been found as follows*.

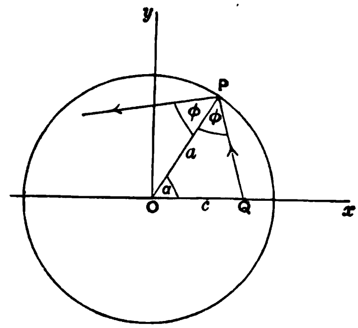


Fig. 93.

Let Q be the radiant point, O the centre, and let $OQ = c$, $OP = a$, and let the angle QOP be α.

Taking the centre as origin and the line OQ as prime vector, we may write the equations of the incident and reflected rays respectively as

$$u = A\cos\theta + B\sin\theta \quad\text{...............................(i)},$$

and $$u = A\cos(2\alpha - \theta) + B\sin(2\alpha - \theta) \quad\text{............ (ii)}.$$

For if we draw equal radii from O to any two points, one on each ray, they must make equal angles with OP, or the sum of their vectorial angles is 2α.

To determine A and B, put $\theta = 0$ and $\theta = \alpha$ in the equation (i) of the incident ray; then

$$1/c = A, \quad 1/a = A\cos\alpha + B\sin\alpha.$$

Hence on substitution the equation (ii) of the reflected ray is

$$u\sin\alpha = \frac{1}{a}\sin(2\alpha - \theta) - \frac{1}{c}\sin(\alpha - \theta) \quad\text{......... (iii)}.$$

Let $2\alpha - \theta = 2\beta$; then this equation may be written as

$$u\sin(\beta + \tfrac{1}{2}\theta) + \frac{1}{c}\sin(\beta - \tfrac{1}{2}\theta) = \frac{1}{a}\sin 2\beta,$$

or as $$U/\cos\beta + V/\sin\beta = 1 \quad\text{.................. (iv)},$$

where $$U = \frac{a}{2}\left(\frac{1}{r} + \frac{1}{c}\right)\cos\tfrac{1}{2}\theta, \quad V = \frac{a}{2}\left(\frac{1}{r} - \frac{1}{c}\right)\sin\tfrac{1}{2}\theta \quad\text{...... (v)}.$$

The envelope of this straight line for all values of α or β is given by differentiating (iv) with regard to β, whence

$$\frac{U}{\cos^3\beta} = \frac{V}{\sin^3\beta} = 1.$$

* Cayley, *Collected Papers*, Vol. II. p. 336—380.

The equation of the caustic is therefore

$$U^{\frac{2}{3}} + V^{\frac{2}{3}} = 1 \text{ .... (vi).}$$

To rationalise the equation, we cube both sides, whence

$$U^2 + V^2 + 3(UV)^{\frac{2}{3}} = 1,$$

and cubing again, the equation is

$$(1 - U^2 - V^2)^3 = 27 U^2 V^2.$$

To transform to Cartesian coordinates, we have from (v),

$$U^2 + V^2 = \frac{a^2}{4}\left(\frac{1}{r^2} + \frac{1}{c^2} + \frac{2}{rc}\cos\theta\right)$$

$$= \frac{a^2}{4}\frac{c^2 + r^2 + 2cx}{c^2 r^2},$$

$$UV = \frac{a^2}{8}\left(\frac{1}{r^2} - \frac{1}{c^2}\right)\sin\theta = \frac{a^2(c^2 - r^2)\,y}{8c^2 r^3}.$$

The Cartesian equation of the caustic is therefore

$$\{(4c^2 - a^2)(x^2 + y^2) - 2a^2cx - a^2c^2\}^3 = 27a^4c^2y^2(x^2 + y^2 - c^2)^2 \text{...(vii).}$$

202. This equation can also be found in another way, which serves to determine the coordinates of any point on the curve in terms of the parameter α, and gives the class of the curve.

The equations of the tangent and normal to the circle at P are

$$x\cos\alpha + y\sin\alpha = a, \text{ and } x\sin\alpha - y\cos\alpha = 0.$$

Hence the equation of the incident ray is

$$(x\cos\alpha + y\sin\alpha - a)/(c\cos\alpha - a) - (x\sin\alpha - y\cos\alpha)/c\sin\alpha = 0,$$

and the equation of the reflected ray is

$$(x\cos\alpha + y\sin\alpha - a)/(c\cos\alpha - a) + (x\sin\alpha - y\cos\alpha)/c\sin\alpha = 0,$$

or

$$x(c\sin 2\alpha - a\sin\alpha) + y(a\cos\alpha - c\cos 2\alpha) = ac\sin\alpha \text{... (i).}$$

On differentiating this equation with regard to α, we obtain

$$x(2c\cos 2\alpha - a\cos\alpha) - y(a\sin\alpha - 2c\sin 2\alpha) = ac\cos\alpha \text{...(ii).}$$

If we put $x = \xi - c$ (*i.e.* transfer the origin to a point on the diameter through the bright point at the same distance from

the centre), we obtain the coordinates of any point on the caustic in the forms

$$\left.\begin{array}{l} \xi\,(a^2 - 3ac\cos\alpha + 2c^2) = 2c^2\,(c - a\cos^3\alpha) \\ y\,(a^2 - 3ac\cos\alpha + 2c^2) = 2ac^2\sin^3\alpha \end{array}\right\}\text{.........(iii).}$$

Again, if in equation (i) we make the substitution $t = \tan\frac{1}{2}\alpha$, the resulting equation is

$$(c + a)\,yt^4 + 4\,\{\tfrac{1}{2}a\,(x + c) + cx\}\,t^3 - 6cyt^2$$
$$+ 4\,\{\tfrac{1}{2}a\,(x + c) - cx\}\,t + (c - a)\,y = 0\text{...(iv),}$$

shewing that through any given point four tangents, real or imaginary, can be drawn to the caustic.

The invariants of this biquadratic in t are

$$I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = (4c^2 - a^2)\,(x^2 + y^2) - 2a^2cx - a^2c^2,$$

$$J \equiv \begin{vmatrix} a_0, & a_1, & a_2 \\ a_1, & a_2, & a_3 \\ a_2, & a_3, & a_4 \end{vmatrix} = a^2cy\,(x^2 + y^2 - c^2),$$

and the discriminant is $I^3 = 27J^2$, the equation of the caustic found above.

The form of this equation shews that there are cusps on the curve at the two points where it meets the axis of x, and also at the two points of intersection with the circle $x^2 + y^2 = c^2$.

It is not difficult to prove, by putting t for $\tan\frac{1}{2}\alpha$ in (iii) and determining unequal values of t, which repeat the values of ξ and y, that the circular points at infinity are also cusps, and that there are four imaginary double points. The degree of the curve is six, there are four double points and six cusps, and its class is four.

203. *To determine the points of intersection of the caustic with the reflecting circle, its asymptotes and cusps.*

(i) By the formula for primary foci, if ϕ be the angle of incidence, and if $QP = u$, $PQ' = v$, we have, $1/u + 1/v = 2/a\cos\phi$.

Hence, if $\phi = \frac{1}{2}\pi$, v will be zero, *i.e.* the caustic meets the circle at the points of contact of tangents from the bright point, and must touch the circle there, since the tangent to the circle being both the incident and the reflected ray coincides with the tangent to the caustic.

The primary focus Q' will also lie on the circle if $v = 2a\cos\phi$, *i.e.* if $u = \frac{2}{3}a\cos\phi$, or QP must be one-third of the chord through Q. In this case therefore Q must lie within the reflector, and since the ratio of QP to the chord through Q lies between $(a-c)/2a$ and $(a+c)/2a$, the necessary conditions that the caustic may cut the circle in real points are $a > c > \frac{1}{3}a$.

(ii) The asymptotes are given by making v infinite, whence $u = \frac{1}{2}a\cos\phi = \frac{1}{4}$ (chord through Q); and therefore by similar reasoning the necessary conditions that real asymptotes exist are $a > c > \frac{1}{2}a$.

Since $c^2 = u^2 - 2ua\cos\phi + a^2$, this relation between u and ϕ gives $\quad \cos\phi = \{4(a^2 - c^2)/3a^2\}^{\frac{1}{2}}$, $\sin\phi = \{(4c^2 - a^2)/3a^2\}^{\frac{1}{2}}$.
But the perpendicular from the centre O on the asymptote is equal to the perpendicular from O on the incident ray, and therefore

$$p = a\sin\phi = \{(4c^2 - a^2)/3\}^{\frac{1}{2}}.$$

The value of u is $\{(a^2 - c^2)/3\}^{\frac{1}{2}}$, and the asymptotes are the rays reflected at the corresponding points of incidence; these lie on the smaller part of the circle cut off by the ordinate through Q.

(iii) To determine the cusps, substitute for $\cos\phi$ in terms of u. We obtain $1/v + 1/u = 4u/(u^2 + a^2 - c^2)$,

whence $$v = \frac{u(u^2 + a^2 - c^2)}{3u^2 - a^2 + c^2}, \quad v + u = \frac{4u^3}{3u^2 - a^2 + c^2}.$$

The arc of the caustic between any two points is the difference of the values of $[v + u]$ for those points.

The cusps are given by

$$d(v+u) = 0, \text{ i.e. by } (u^2 - a^2 + c^2)u^2\,du = 0.$$

The factor $du = 0$ indicates that the ray is incident at either end of the diameter through Q; there are therefore two cusps on the axis of symmetry.

The other factor $u^2 - a^2 + c^2 = 0$ shews that Q must lie within the circle, and that the incident ray is perpendicular to OQ; also this value of u^2 gives $v = u$, *i.e.* these cusps lie on the circle of radius c through the bright point, and the caustic touches that

circle, since the perpendicular from the origin on both the incident and reflected rays is c.

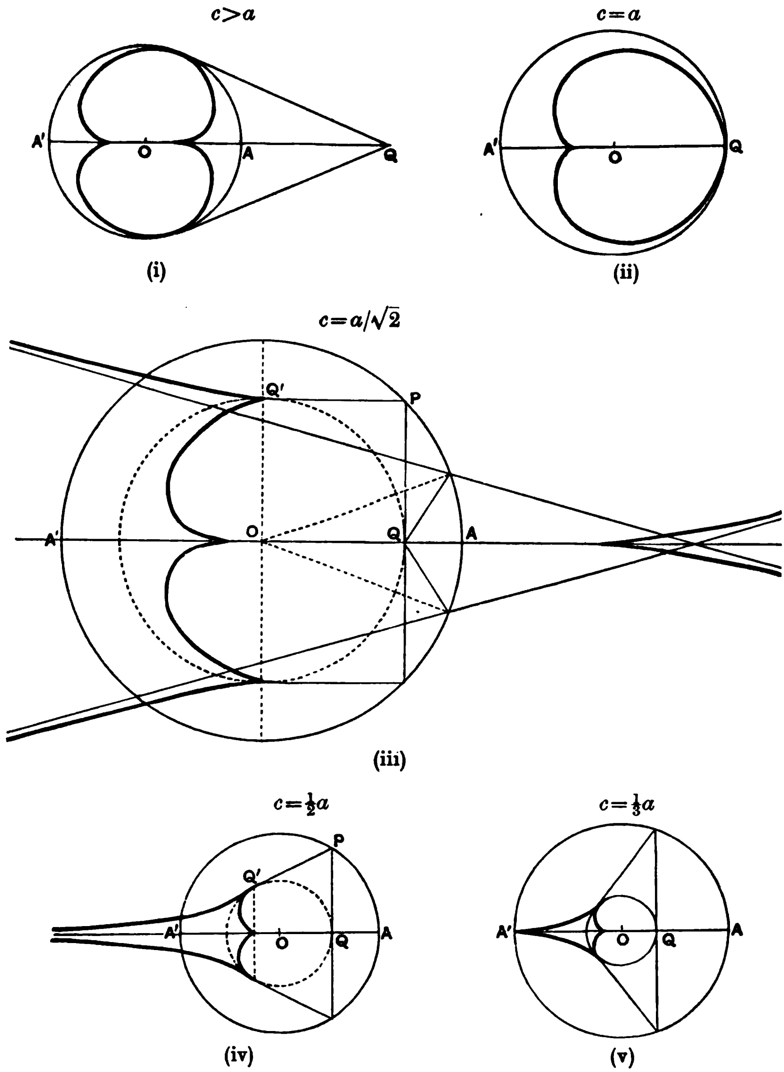


Fig. 94.

When $c>a$, the caustic is entirely within the mirror (fig. i), and is not unlike the two-cusped epicycloid, which is its shape when c is infinite. When $c=a$ the caustic is a cardioid (fig. ii). When c/a is between 1 and $\frac{1}{2}$, the curve has two asymptotes, two

cusps on the axis of x and two cusps on the circle through Q (fig. iii). When $c = \frac{1}{2}a$, the two asymptotes coincide with the axis of x (fig. iv). When $c < \frac{1}{2}a$, the caustic consists of a single finite branch with four cusps, and lies entirely within the mirror if $c < \frac{1}{3}a$, having a cusp on the mirror if $c = \frac{1}{3}a$ (fig. v).

Part at least of these curves may be seen when rays from a bright point are reflected at a circular cylinder; since an orthotomic surface of the light is the limaçon in the cross-section through the origin of light, and all the evolutes of this curve are the above caustic curve repeated in parallel planes.

204. Caustic at a circle after n reflections.

In two cases, (i) when the radiant point is on the circle, (ii) when it is at infinity, we can determine geometrically the orthotomic curves and the caustics for rays reflected at the circle n times.

(i) If Q be the radiant point, $P_1, P_2 \ldots P_n$ the points of reflection (fig. 95), and ϕ the angle of incidence,

$$QP_1 + P_1P_2 + \ldots + P_{n-1}P_n = 2na \cos \phi.$$

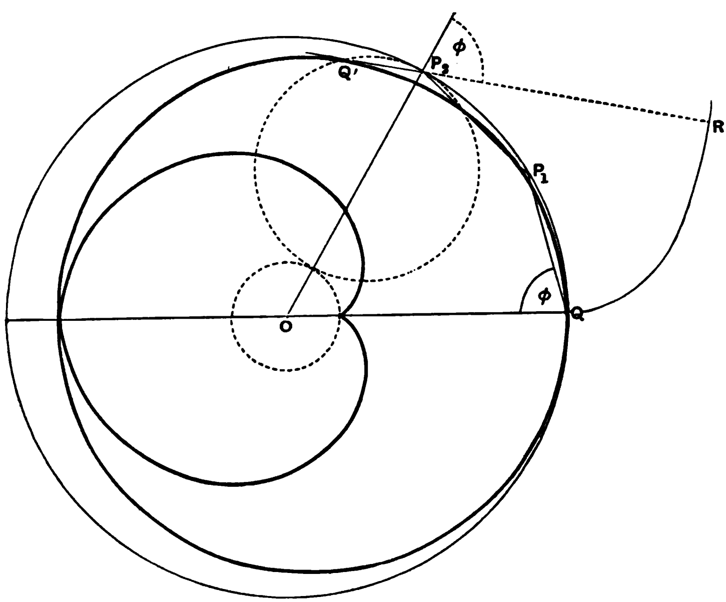


Fig. 95.

Hence an orthotomic curve is found by taking P_nR, equal to $2na\cos\phi$, opposite to the final direction of the ray. Also P_nR makes an angle ϕ with OP_n produced, and is therefore a chord of a circle of radius na touching the fixed circle externally. The arc P_nR of this circle $= na(\pi - 2\phi) =$ the arc QP_n of the fixed circle.

The orthotomic curve is therefore the epicycloid traced out by a point on a circle of radius na rolling on a circle of radius a; its evolute, the caustic, is a similar epicycloid in which a circle of radius $\frac{na}{2n+1}$ rolls on a circle of radius $\frac{a}{2n+1}$. The cusp of the orthotomic epicycloid and the vertex of the caustic fall at the bright point Q.

The caustic is drawn in the figure for the case of $n = 2$; rays from Q twice reflected at the upper half of the mirror touch the caustic in points between the vertex and the cusp.

Epicycloids of this nature, in which the radius of the rolling circle is a multiple of that of the fixed circle, will have their cusps and vertices on the same axis, and will also have double points, both on and off the axis.

(ii) If the incident rays be perpendicular to the diameter BOB' (fig. 96), we take this diameter as an initial orthotomic curve. Then

$$MP_1 + P_1P_2 + \ldots + P_{n-1}P_n = (2n-1)\,a\cos\phi\,;$$

and therefore an orthotomic curve is found by taking R on the final ray at distance $(2n-1)\,a\cos\phi$ backwards from P_n.

Hence P_nR is a chord of a circle of radius $(n-\frac{1}{2})\,a$, cutting off from that circle an arc of length $(n-\frac{1}{2})\,a\,(\pi - 2\phi)$, which is equal to the arc BP_n of the reflector.

One orthotomic curve is therefore an epicycloid in which a circle of radius $(n-\frac{1}{2})\,a$ rolls on a circle of radius a; and the radii of the rolling and fixed circles in its evolute are $\frac{2n-1}{4n}a$ and $\frac{a}{2n}$ respectively.

The caustic has vertices at B and B', and cusps on the axis AA'; the figure is drawn for the case $n = 2$.

When the rays emanate from any other point, the chief features of the caustic may be determined analytically as in the following article.

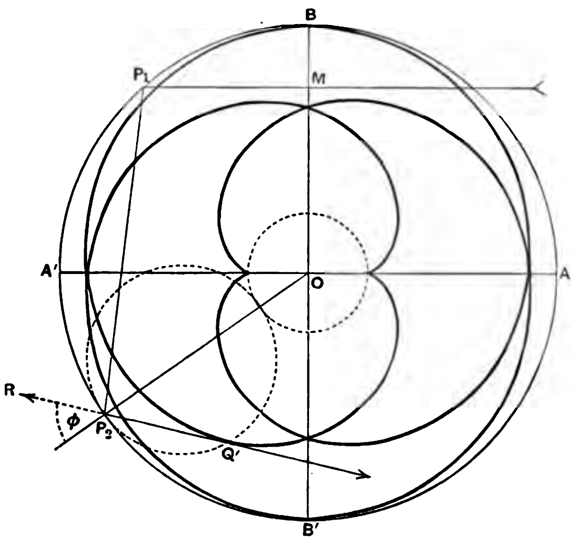


Fig. 96.

205. *Rays from a point are reflected n times at a circle; to determine the cusps and asymptotes of the caustic.*

Let the ray QP_1 make an angle θ with the axis of x, and let ϕ be the angle of incidence at P_1. The deviation at each reflection being $\pi - 2\phi$, the ray after n reflections makes an angle $\theta + n(\pi - 2\phi)$ with the axis of x. Also the perpendicular from the centre O on the ray is $a \sin \phi$ throughout. Hence the equation of the ray after n reflections is

$$x \sin(2n\phi - \theta) + y \cos(2n\phi - \theta) = (-)^{n-1} a \sin \phi \quad \text{............(i).}$$

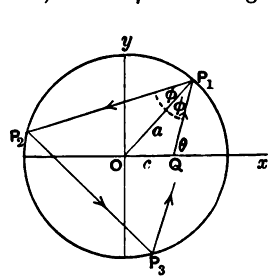


Fig. 97.

The angles θ and ϕ are connected by the equation $c \sin \theta = a \sin \phi$, whence $\cot \theta \, d\theta = \cot \phi \, d\phi$, and therefore differentiating (i), the envelope is found from it and the equation

$$x \cos(2n\phi - \theta) - y \sin(2n\phi - \theta) = (-)^{n-1} \frac{a \cos \phi \tan \phi}{2n \tan \phi - \tan \theta} \quad \text{...... (ii).}$$

Let $2n \tan \phi - \tan \theta = \tan \omega$; then solving these equations we obtain the coordinates of any point on the caustic in the form

$$\left.\begin{array}{l} x = (-)^{n-1} a \sin \phi \cos(2n\phi - \theta - \omega) \operatorname{cosec} \omega \\ y = \;\; -)^{n-1} a \sin \phi \sin(2n\phi - \theta - \omega) \operatorname{cosec} \omega \end{array}\right\} \quad \text{............ (iii).}$$

If Q' be this point, and P_n the last point of incidence, the vectorial angle of P_n is $\theta-\phi+(n-1)(\pi-2\phi)$; hence if x', y' be its coordinates,

$$x'=(-)^{n-1}a\cos\{(2n-1)\phi-\theta\}=(-)^{n-1}a\{\cos(2n\phi-\theta)\cos\phi +\sin(2n\phi-\theta)\sin\phi\},$$

and therefore $\quad x-x'=(-)^{n-1}a\cos(2n\phi-\theta)(\sin\phi\cot\omega-\cos\phi).$

Similarly $\quad y-y'=(-)^{n-1}a\sin(2n\phi-\theta)(\sin\phi\cot\omega-\cos\phi).$

But the inclination of P_nQ' to the axis of x is $\theta+n(\pi-2\phi)$, and therefore

$$P_nQ'=a(\cos\phi-\sin\phi\cot\omega)\quad\text{(iv)}.$$

(i) *Intersections with the reflector.* Since from (iii)

$$x^2+y^2=a^2\sin^2\phi\operatorname{cosec}^2\omega,$$

the caustic meets the circle at the points given by $\sin\phi=\pm\sin\omega$.

This equation, with the one above defining ω, is satisfied by $\phi=\omega=\frac{1}{2}\pi$, indicating the points of contact of tangents from the bright point; and by $\tan\phi=\pm\tan\omega$, whence $(2n\pm1)\tan\phi=\tan\theta$. We deduce the equation

$$\tan^2\theta=\{(2n\pm1)^2c^2-a^2\}/(a^2-c^2).$$

Consecutive caustics have therefore a common point on the circle; and the conditions that the nth caustic may meet the circle in four real points are

$$a>c>a/(2n\pm1).$$

(ii) *Asymptotes.* These are given by $2n\tan\phi=\tan\theta$; whence

$$\tan^2\theta=(4n^2c^2-a^2)/(a^2-c^2),$$

and there are therefore real asymptotes only if $a>c>a/2n$.

The perpendicular from O on the two asymptotes is equal to $a\sin\phi$, *i.e.* to

$$\{(4n^2c^2-a^2)/(4n^2-1)\}^{\frac{1}{2}}.$$

(iii) *Arc of the caustic.* An orthotomic curve is given by the equation $QP_1+P_1P_2+\ldots+P_nR=\text{constant}$; or taking the constant zero,

$$RP_n=a\sin(\theta-\phi)\operatorname{cosec}\theta+(n-1)2a\cos\phi.$$

Hence
$$\begin{aligned}RQ'&=a\{\cos\phi-\sin\phi\cot\theta+2(n-1)\cos\phi+\cos\phi-\sin\phi\cot\omega\}\\&=a\{2n\cos\phi-\sin\phi(\cot\theta+\cot\omega)\}\\&=a\left\{2n\cos\phi-\frac{2n\sin\phi\tan\phi\cot\theta}{2n\tan\phi-\tan\theta}\right\}\\&=2n\left\{a\cos\phi-\frac{c^2\cos^2\theta}{2nc\cos\theta-a\cos\phi}\right\}\quad\text{(v)}.\end{aligned}$$

If this be differentiated with regard to θ, using the relation between θ and ϕ, we obtain

$$\frac{d}{d\theta}(RQ')=-\frac{2nc^2\sin\theta\cos\theta}{\cos\phi}\left\{\frac{3(nc\cos\theta-a\cos\phi)^2+(n^2-1)c^2\cos^2\theta}{(2nc\cos\theta-a\cos\phi)^2}\right\}\ldots\text{(vi)}.$$

The tangent to the caustic at Q' makes an angle $(2n\phi-\theta)$ with a fixed line, and

$$\frac{d}{d\theta}(2n\phi-\theta)=\frac{2nc\cos\theta-a\cos\phi}{a\cos\phi}\quad\text{(vii)}.$$

If we divide the expression in (vi) by that in (vii), we have the radius of curvature of the caustic at any point.

The cusps on the caustic are given from (vi), either by the equation $\sin\theta=0$, *i.e.* they lie on the axis; or by the equation $\cos\theta=0$, *i.e.* the rays emanating from Q perpendicular to the axis give rise to cusps on all the caustics, and the cusps all lie on the circle $x^2+y^2=c^2$.

CAUSTIC BY REFRACTION.

206. When rays from a point Q are refracted at a given curve from a medium of index μ into one of index μ', the caustic is the evolute of the family of orthotomic curves given by the equation

$$\mu\, QP + \mu'\, PR = c,$$

P being the point of incidence. The simplest form of the orthotomic curve may be derived by taking R on the refracted ray behind the point of incidence so that $RP = \frac{\mu}{\mu'} QP$, c being then zero.

The distance from P of the primary focus Q', which is the point of contact of the ray with the caustic, is given by the formula

$$\frac{\mu' \cos^2 \phi'}{v} - \frac{\mu \cos^2 \phi}{u} = \frac{\mu' \cos \phi' - \mu \cos \phi}{\rho}.$$

Hence RQ' and the arc of the caustic may be found.

207. Caustic by refraction at a plane surface.

Let Q be the origin of light, QP a ray incident at P on a plane refracting surface, and RP the direction of the refracted ray.

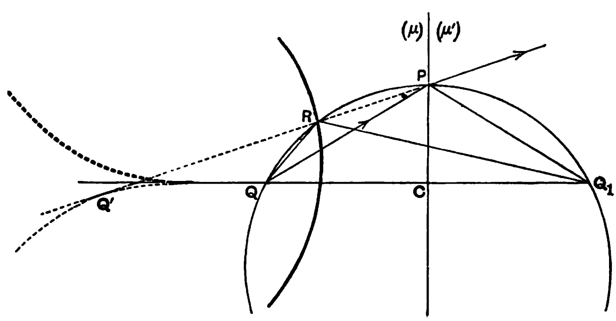


Fig. 98.

To determine an orthotomic curve, take R so that

$$RP/QP = \mu/\mu' = \sin \phi'/\sin \phi,$$

where ϕ and ϕ' are the angles of incidence and refraction. In the triangle QPR the angle QPR is $\phi - \phi'$; it follows that the angles QRP and RQP are respectively $\pi - \phi$ and ϕ'. The circle QRP therefore touches the normal to the plane at P.

If this circle cut the normal to the plane through Q again in the point Q_1, we have by Ptolemy's theorem

$$QR \cdot Q_1P + RP \cdot QQ_1 = QP \cdot Q_1R.$$

Divide by QP or Q_1P, and substitute for RP, we obtain for the locus of R the equation

$$Q_1R - QR = \frac{\mu}{\mu'} QQ_1.$$

The orthotomic curve is therefore one branch of a hyperbola, with foci at Q and Q_1, and of eccentricity μ'/μ.

If the ray be refracted into a rarer medium, the point R will fall on the circle below QQ_1, and the same argument gives for the locus of R the equation

$$QR + Q_1R = \frac{\mu}{\mu'} QQ_1.$$

The orthotomic curve is therefore in this case an ellipse, and the caustic is its evolute, the cusps on the refracting line being the points of incidence of rays that are refracted along the line of separation of the media (cf. Art. 168).

208. Caustic by refraction at a circle.

The caustic by refraction at a circle for rays emanating from a point is the evolute of parts of a Cartesian oval.

Let the rays from a point Q be refracted at a circle, centre O, into a medium of relative refractive index μ, where we suppose $\mu > 1$; also let the distance $OQ = c$, and let the radius $OA = a$.

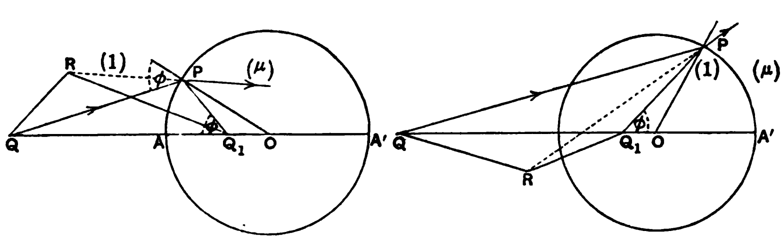


Fig. 99.

(i) Let QP be a ray incident on the part of the circle convex to Q, and let R be taken on the refracted ray backwards so that $RP = QP/\mu$ (fig. i). The locus of R is an orthotomic curve.

Since the angle QPR is $\phi - \phi'$, and the ratio of the sides of the triangle QPR which contain this angle is $\sin\phi : \sin\phi'$, it follows that the angle PQR is ϕ', and the angle PRQ is $\pi - \phi$.

Again, let Q_1 be the point inverse to Q in the circle; the triangles OPQ, OQ_1P are similar, and therefore the angle PQ_1Q is ϕ. Hence PQ_1QR is a cyclic quadrilateral, and we have by Ptolemy's theorem

$$QR \,.\, Q_1P + RP \,.\, QQ_1 = QP \,.\, Q_1R.$$

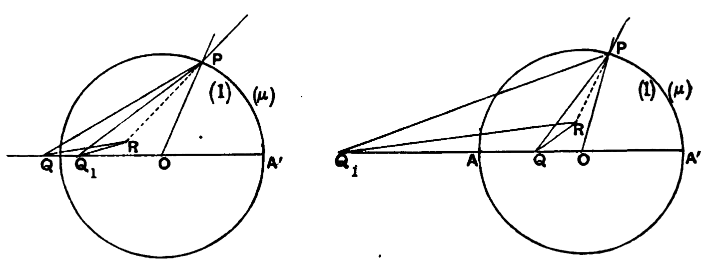
Since $Q_1P/QP = a/c$, and $RP/QP = 1/\mu$, the locus of R is therefore part of the Cartesian oval whose equation is

$$Q_1R - \frac{a}{c}QR = \frac{QQ_1}{\mu} \quad \text{.........................(i).}$$

(ii) For rays from Q incident on the part of the circle concave to Q, there is a critical case when $OQ = \mu OA$. All the rays refracted at the concave part of the circle then pass accurately through Q_1, while the caustic for rays incident on the convex part is part of the caustic by reflection for rays from Q_1.

But if $c > \mu a$, and R be taken on the refracted ray behind P as before, we find that PQ_1RQ is a cyclic quadrilateral (fig. ii), and that the locus of R is part of the oval whose equation is

$$Q_1R + \frac{a}{c}QR = \frac{QQ_1}{\mu} \quad \text{.....................(ii).}$$



(iii) Fig. 99. (iv)

If however $c < \mu a$, PR falls to the right of PQ_1 (fig. iii), and the locus of R is given by the equation

$$\frac{a}{c}QR - Q_1R = \frac{QQ_1}{\mu} \quad \text{.....................(iii).}$$

If we express the actual distances QR, Q_1R in terms of the Cartesian coordinates of R, the equations (i) and (ii) when rationalised lead to the same equation.

The caustic of the refracted rays is nevertheless only the parts of the evolute corresponding to the parts of either of the ovals determined in each case.

(iii) Let Q be within the circle (fig. iv); the quadrilateral $PRQQ_1$ is cyclic as before, and the locus of R is the oval whose equation is

$$Q_1R - \frac{a}{c}QR = \frac{QQ_1}{\mu} \qquad \text{(iv)}.$$

The whole of this branch of a certain Cartesian oval is orthotomic to the emergent rays, and the caustic consists entirely of its evolute.

209. A Cartesian oval has three foci; and its equation can be written in three ways in the form $lr + mr' = $ constant. Cayley has shewn that the same oval can be the orthotomic curve for six arrangements of the bright point, the radius of the circle, and the refractive index.

It is easy to verify on rationalising equations (i) or (ii) that the same equation is obtained if for μ we substitute $1/\mu$, for c, c/μ^2, and for a, a/μ, the centre of the circle remaining the same point. If Q_2 denote the point at distance c/μ^2 from O, Q_1 and Q_2 are images in the circle of radius a/μ. The orthotomic curves are therefore of the same type whether the index of refraction be greater or less than unity, Q and Q_2 being the corresponding radiant points.

The parts of the circles which give rise to the orthotomic curves are however different.

(i) When $c > \mu a$, Q_2 is outside the smaller circle; and it will be found that the rays from Q incident on the convex part of the circle a give the same oval as the rays from Q_2 incident on the concave part of the circle a/μ; and conversely.

(ii) When $c < \mu a$, Q_2 is inside the smaller circle, and the ovals orthotomic to rays from Q incident on the convex or concave parts are the ovals orthotomic to rays from Q_2, which emerge near A' or A respectively. Certain parts of these ovals are lacking, as the rays from Q_2 may suffer internal reflection.

(iii) Lastly, when Q is within the circle, Q_2 is also within the smaller circle; and it will be found that the orthotomic curve for rays from Q_2 is the oval $Q_1R + \frac{\mu a}{c}Q_2R = \mu Q_1Q_2$, which is not the oval given in equation (iv), but the other branch of the same Cartesian.

210. Caustic by refraction for rays from a point on the circle.

The construction given in Art. 208 fails when the bright point is on the circle; in this case the caustic by refraction is the same as a certain caustic by reflection.

Using the same construction for the orthotomic curve as before, we find that the angle PRQ is $\pi - \phi$.

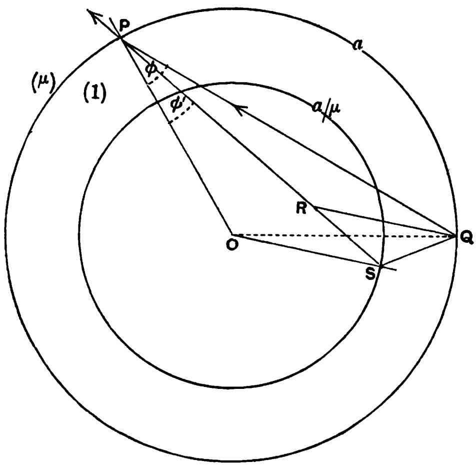


Fig. 100.

Describe a circle of radius a/μ and let PR cut it in S; we have $\sin OSP/\sin \phi' = OP/OS = \mu$, and therefore the angle OSP is ϕ, and is equal to the angle OQP. Hence the points $OPQS$ are concyclic, and the angle QSO is $\pi - \phi$. It follows that QS and SR make equal angles with OS, that QR and SO are parallel, and that $SR = SQ$.

Hence the locus of R is the orthotomic curve for rays from Q reflected at the inner circle; and the caustic by refraction for rays from a point on a circle of radius a is the caustic by reflection for rays from that point at a circle of radius a/μ. This caustic however extends only from the cusp on the axis between O and Q to the points in which it touches the smaller circle (cf. Fig. i, Art. 203).

If the relative index of refraction be less than unity, the same construction will hold, but the origin Q will be inside the circle of

radius a/μ, and the part of the caustic by reflection which arises from the larger arc of the circle cut off by the chord through Q perpendicular to OQ will be lacking, since the corresponding refracted rays do not exist, the angles of incidence exceeding the critical angle. The caustic extends from the cusps on the smaller circle to the outer cusp on the axis, and there are two asymptotes if $\mu > \frac{1}{2}$ (cf. Figs. iii, iv, v, Art. 203).

211. Caustic for parallel rays refracted at a circle.

The centre being the origin, and the incident rays being in the direction of the axis of x, the equation of the refracted ray is

$$x \sin(\phi - \phi') + y \cos(\phi - \phi') = a \sin \phi' \quad \text{..........(i)}.$$

Since $\sin \phi = \mu \sin \phi'$, and $d\phi/\tan\phi = d\phi'/\tan\phi'$, the envelope is given on differentiating (i) by the equation

$$x \cos(\phi - \phi') + y \sin(\phi - \phi') = \frac{a \cos\phi' \tan\phi'}{\tan\phi - \tan\phi'}$$

$$= \frac{a \sin\phi' \cos\phi \cos\phi'}{\sin(\phi - \phi')} \quad \text{.......... (ii)}.$$

Eliminating x, we obtain

$$y = a \sin\phi' \{\cos(\phi - \phi') - \cos\phi \cos\phi'\}$$

$$= a \sin\phi \sin^2\phi' = \frac{a}{\mu^2} \sin^3\phi \quad \text{..................... (iii)}.$$

If we substitute in (i) and multiply by $\sin(\phi + \phi')$, we have

$$x(\sin^2\phi - \sin^2\phi') = a \sin\phi' \{1 - \sin\phi \sin\phi' \cos(\phi - \phi')\} \sin(\phi + \phi')$$

$$= a \sin\phi' \{\sin(\phi + \phi') - \sin\phi \sin\phi' (\sin\phi \cos\phi + \sin\phi' \cos\phi')\}$$

$$= a \sin\phi' \{\sin\phi \cos^3\phi' + \sin\phi' \cos^3\phi\}$$

or $$x(\mu^2 - 1) = a(\mu \cos^3\phi' + \cos^3\phi) \quad \text{.....................(iv)}.$$

Hence the caustic, given by (iii) and (iv), is

$$x(\mu^2 - 1) = (\mu^{\frac{2}{3}} a^{\frac{2}{3}} - y^{\frac{2}{3}})^{\frac{3}{2}} + (a^{\frac{2}{3}} - \mu^{\frac{4}{3}} y^{\frac{2}{3}})^{\frac{3}{2}}. \quad \text{........(v)}.$$

If in (iii) and (iv) for μ we put $1/\mu$, for a, a/μ, and exchange ϕ and ϕ', the coordinate y remains the same while x changes sign. On rationalising (v) the change of sign disappears, and therefore the caustic by refraction for parallel rays at a circle of radius a and index μ is the same as that for a concentric circle of radius a/μ and index $1/\mu$.

The curve is composed of two parts and is symmetrical about both axes. The part drawn in the figure answers to rays refracted

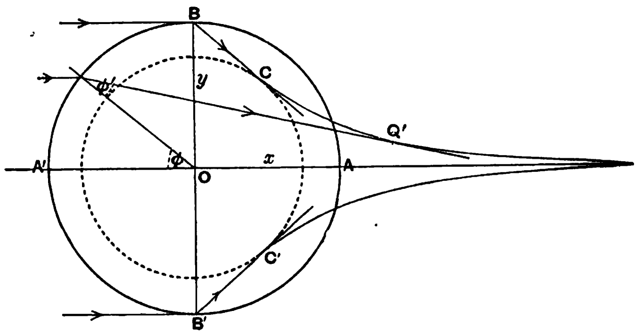


Fig. 101.

at the convex half of the circle, the other part to rays refracted at the concave half. There are two cusps on the axis of x, and the curve ceases abruptly on the circle of radius a/μ at the points on the caustic answering to rays, which are incident on the circle at grazing incidence, and are refracted at the critical angle. Or in the second mode of construction of the caustic they answer to the rays incident on the smaller circle at the critical angle and refracted along the tangent. The figure is drawn for $\mu = 4/3$.

THE RAINBOW.

212. The rainbow consists of a brightly coloured arc, seen in the part of the sky opposite to the sun, when its rays shine upon falling rain. The inner part of the circle is violet, the outer red, and between them are the same colours as those seen in Newton's experiment with a prism. The centre of the circle is always the point opposite to the sun, and the arc visible depends upon the sun's altitude, being greatest when the sun is near the horizon. Not unfrequently a second larger circle, also coloured, is seen outside the first. In this the red is innermost and the violet outermost; while the space between the bows is distinctly darker than the rest of the cloud.

A similar appearance may sometimes be seen when the sun is shining upon dew on the ground, especially when the observer

is at a considerable elevation. The appearance presented may be a coloured ellipse or hyperbola, being the section by the horizontal plane of a cone with its vertex at the observer.

The general explanation is that of all the parallel rays of light which fall on a drop of water and emerge after one or more internal reflections, those which emerge in appreciably the same direction reinforce each other, and therefore produce a definite sensation on the eye. The order of the colours is explained by the fact that the direction of these efficient rays depends on the index of refraction of the drop, and is therefore different for different colours.

This explanation is not entirely satisfactory, nor are the results absolutely consistent with the facts. The rays that leave the drop in the same direction take slightly different paths in the drop, and may therefore be in a condition to produce interference effects in accordance with the principles of Physical Optics. We shall however determine the minimum deviation, and the general form of the caustic surface enveloped by a system of rays, originally parallel, and emerging after any number of reflections within a drop of water.

213. Let a pencil of parallel rays, incident on a sphere of water, be internally reflected n times, and finally refracted into air again. The path of any ray will lie throughout in the initial plane of incidence, and all the emergent rays will touch a surface of revolution about the diameter in the common direction of the incident rays.

Let ϕ be the angle of incidence of any ray, ϕ' that of refraction; the deviation is made up of $\phi - \phi'$ at the first refraction, $\pi - 2\phi'$ at each reflection, and $\phi - \phi'$ at emergence, all measured in the same direction.

Hence $$D = n\pi + 2\{\phi - (n+1)\phi'\} \qquad \text{(i)}.$$

Consecutive emergent rays will be parallel, and the meridian caustic curve will have an asymptote, when the deviation is stationary in value. Since $\sin\phi = \mu \sin\phi'$, we obtain on differentiation

$$\frac{dD}{d\phi} = 2\left\{1 - \frac{(n+1)\cos\phi}{\mu\cos\phi'}\right\} \qquad \text{(ii)}.$$

Hence D is stationary in value when $\mu \cos\phi' = (n+1)\cos\phi$, *i.e.* $\mu^2 - \sin^2\phi = (n+1)^2 \cos^2\phi$; and therefore

$$\sin^2\phi = \frac{(n+1)^2 - \mu^2}{n(n+2)}, \quad \cos^2\phi = \frac{\mu^2 - 1}{n(n+2)} \text{ .... (iii).}$$

Since $\mu > 1$, this value of ϕ is possible only if $n > 0$; the caustic formed by rays which pass through without internal reflection therefore has not an asymptote.

If n be unity or any other integer, the deviation given by the value of ϕ obtained in (iii) will be a minimum.

For as ϕ varies from 0 to $\frac{1}{2}\pi$ the differential coefficient $\dfrac{dD}{d\phi}$ vanishes once only, and since, when $\phi = 0$, it is negative, μ being less than $n+1$, the value of D decreases from $n\pi$ to a minimum, and then increases till at $\phi = \frac{1}{2}\pi$ it attains its extreme value.

The value of ϕ given by (iii) and the corresponding values of ϕ' and D involve the index of refraction μ, and therefore the effective rainbow rays emerge in different directions for different colours. Treating μ as variable, we have

$$\frac{\partial D}{\partial \mu} = 2\left\{\frac{\partial \phi}{\partial \mu} - (n+1)\frac{\partial \phi'}{\partial \mu}\right\},$$

where $$\cos\phi \frac{\partial \phi}{\partial \mu} = \mu \cos\phi' \frac{\partial \phi'}{\partial \mu} + \sin\phi'.$$

Hence

$$\frac{\partial D}{\partial \mu} = 2\left[\left\{\frac{\mu\cos\phi'}{\cos\phi} - (n+1)\right\}\frac{\partial \phi'}{\partial \mu} + \frac{\sin\phi'}{\cos\phi}\right],$$

and since for the effective rays the coefficient of $\dfrac{\partial \phi'}{\partial \mu}$ vanishes,

$$\left(\frac{\partial D}{\partial \mu}\right) = \frac{2}{\mu}\tan\phi.$$

The minimum deviation therefore increases throughout the spectrum from the red to the violet rays.

214. The coordinates of any point on the caustic of the emergent rays may be determined as follows.

Taking the centre of the sphere as origin, let the negative direction of the axis of x be that of the incident rays. The emergent ray therefore, which is incident at angle ϕ, makes an angle $\pi + D\,(=\theta)$ with the axis of x; and the perpendicular from

the centre on the incident and emergent rays are both $a \sin \phi$, the perpendicular on the ray during its path in the drop being $a \sin \phi'$ throughout. Hence the equation of the emergent ray is

$$x \sin \theta - y \cos \theta = a \sin \phi \quad \text{.............................(i).}$$

For the envelope we obtain, on differentiating, the equation

$$x \cos \theta + y \sin \theta = \frac{\mu}{2} \frac{a \cos \phi \cos \phi'}{\mu \cos \phi' - (n+1) \cos \phi} \quad \text{........(ii).}$$

The vectorial angle of the point of emergence is

$$\phi + (n+1)(\pi - 2\phi') \text{ or } \theta - \phi.$$

If we move the origin to this point, these equations become

$$\xi \sin \theta - \eta \cos \theta = 0 \quad \text{.......................(iii),}$$

$$\xi \cos \theta + \eta \sin \theta + a \cos \phi = \frac{\mu}{2} \frac{a \cos \phi \cos \phi'}{\mu \cos \phi' - (n+1) \cos \phi} \quad \text{...(iv).}$$

Solving these equations, we find that the primary focus, which is the point of contact with the caustic, is at a distance

$$\tfrac{1}{2} a \cos \phi \frac{(2n+2) \cos \phi - \mu \cos \phi'}{(n+1) \cos \phi - \mu \cos \phi'} \quad \text{..............(v),}$$

taken on the emergent ray behind the point of emergence.

The secondary focus throughout lies on the axis of x, the caustic surface being one of revolution about that axis.

215. Primary Bow.

If the rays suffer one internal reflection, the deviation is a minimum when the ray is incident at an angle $\sin^{-1} \{(4 - \mu^2)/3\}^{\frac{1}{2}}$.

If we take $\mu = 1{\cdot}331$ for red rays, we find that the corresponding angles are

$$\phi = 59° \, 31' \, 21'', \quad \phi' = 40° \, 21' \, 24'', \quad D_R = 137° \, 37' \, 6'';$$

and with $\mu = 1{\cdot}344$ for violet rays, we have

$$\phi = 58° \, 46' \, 20'', \quad \phi' = 39° \, 30' \, 45'', \quad D_V = 139° \, 29' \, 40''.$$

Now if a line be drawn through the eye parallel to the direction of the sun's rays, all drops, which lie on a cone of semi-vertical angle $\pi - D$ with this line as axis, will be in a position to allow the emergent parallel rays to enter the eye. The appearance seen is therefore composed of arcs of different colours; and the angular radii will be 42° 22′ 54″ for the red arc,

and $40° 30' 20''$ for the violet arc. Since however each point of the sun's disc sends rays giving rise to a bow, the apparent breadth of the bow exceeds the difference of these radii by the sun's angular diameter, *i.e.* its breadth

$$= 1° 52' 34'' + 32' = 2° 24' 34''.$$

Moreover as the various bows overlap, the colours seen are pure only at the edges of the band of colours.

Drops of rain which lie within the cone of the rainbow can send some light to the eye, since the inclination of the line of sight to the axis of the cone is less than $\pi - D$, where D is the minimum value. Drops outside this cone cannot send to the eye light that has suffered one internal reflection.

216. To consider the general form of the caustic, we take for simplicity $\mu = \frac{4}{3}$; then for minimum deviation $\phi = 59° 24'$, $\phi' = 40° 12'$, $D = 138°$; while the vectorial angle of the point of internal reflection is $180° + \phi - 2\phi'$, or $159°$, and of emergence is $\phi - 4\phi'$, or $-101° 24'$.

For any ray incident at angle ϕ the distance of the point of contact with the caustic behind the point of emergence is

$$\tfrac{1}{2}a \cos\phi\,(4\cos\phi - \mu\cos\phi')/(2\cos\phi - \mu\cos\phi').$$

For $\phi = 0$, this is $2a$, and the distance remains positive till it becomes infinite. The caustic therefore for rays incident at an angle less than $59° 24'$ lies entirely behind the drop, having a cusp on the drop itself.

After passing the rainbow ray, the expression for the distance changes sign, and the caustic is in front of the drop till the angle ϕ is reached, for which $4\cos\phi = \mu\cos\phi'$. This gives $\phi = 76° 50'$, $\phi' = 46° 55'$, $D = 146°$, and the vectorial angle of the point of emergence is $-110° 50'$.

After this the caustic is within the drop, till finally the ray is incident tangentially, when $\phi = 90°$, $\phi' = 48° 30'$, $D = 166°$, and the angle of the point of emergence is $-103°$; at which point the caustic ends, touching the drop internally as the ray emerges tangentially. It will be found that it is impossible to draw a curve of continuous curvature to touch the emergent rays at the two points last obtained; and in fact the caustic has a cusp at a

point within the drop, the corresponding value of ϕ being just over 78° (cf. Art. 221).

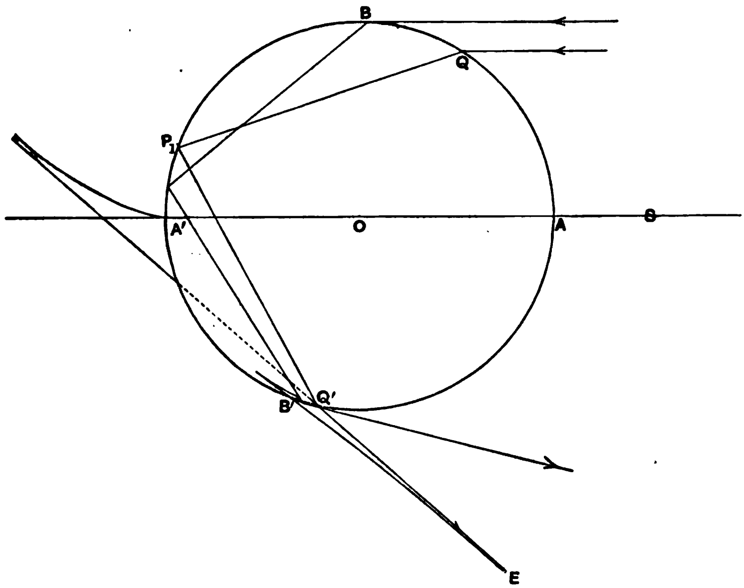


Fig. 102.

The rays incident on a considerable portion of the surface of the drop near the ray of minimum deviation emerge appreciably in the same direction. Taking the limit of visibility as 1′, the angle of incidence may vary fully 30′ on either side of its value for the efficient ray before the deviation exceeds its minimum value by that limit. It is plain then that a considerable amount of light can enter the eye, and the rainbow in strong sunlight appears really very bright.

217. Secondary Bow.

When the rays are reflected twice within the drop, we have for the ray of minimum deviation $\sin\phi = \{(9-\mu^2)/8\}^{\frac{1}{2}}$ and $D = 2\pi + 2\phi - 6\phi'$. With $\mu = 1\cdot331$ for red rays, these values give

$$\phi = 71°\ 54'\ 19'', \quad \phi' = 45°\ 34'\ 20'', \quad D = 230°\ 22'\ 36'';$$

and with $\mu = 1\cdot344$ for violet rays,

$$\phi = 71°\ 29'\ 22'', \quad \phi' = 44°\ 52'\ 26'', \quad D = 233°\ 44'\ 8''.$$

The rays which enter the eye are those which are incident on the lower half of the drop; and the angular radius of the bow is

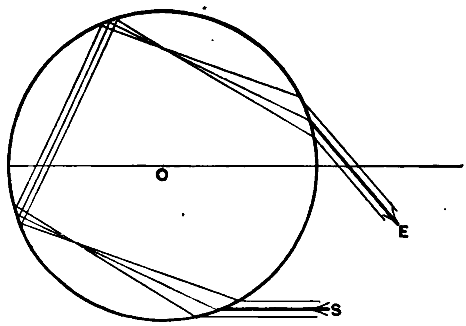


Fig. 103.

$D-\pi$, *i.e.* $50^\circ\, 22'\, 36''$ for the red rays, and $53^\circ\, 44'\, 8''$ for the violet. The red is therefore the innermost and the violet the outermost colour in this bow; also drops that lie without the bow can send some rays to the eye, but drops within the bow can send none. The part of the cloud between the primary and secondary bows can therefore send no light to the eye by one or two reflections, and appears darker than the rest of the cloud.

The breadth of the secondary bow is $3^\circ\, 53'\, 22''$; and it is much fainter than the primary for two reasons:—(i) the intensity of the light is enfeebled at each reflection, and (ii) the relative changes in ϕ and ϕ' proceeding much faster at the larger angle of incidence of the effective rays, these rays emerge from a smaller annulus of the drop than the rays of the primary bow.

218. Considering the caustic of the rays with two internal reflections, the cusp on the axis is at distance $7a/5$ behind the point of emergence, and therefore within the drop; and the caustic lies within the drop till a value of ϕ equal to $59^\circ\, 24'$ is reached, when the deviation has been reduced from 360° to $237^\circ\, 36'$, the vectorial angle of the point of emergence of the ray being $358^\circ\, 12'$, and of the point on the drop where it touches the caustic 277°.

With $\mu=\frac{4}{3}$ we have for the ray of minimum deviation $\phi=71^\circ\, 50'$, $\phi'=45^\circ\, 27'$, $D=230^\circ\, 58'$, and the vectorial angle of the point of emergence of the ray is $339^\circ\, 8'$. The caustic then passes in front of the asymptote, and crosses within the circle at a point given by

$6 \cos \phi = \mu \cos \phi'$; whence $\phi = 81° 26'$, $\phi' = 47° 52'$, $D = 235° 40'$, and the vectorial angle of this point is $334° 30'$.

After running to a cusp within the circle, the caustic finally touches the circle at a point of vectorial angle 339°, the deviation then being 249°.

It will be seen from the numbers above that it is hardly possible to give an accurate drawing of this caustic.

219. Bows of Higher Orders.

These can only be seen under very exceptional circumstances, or by the aid of experiment.

For the effective rays after three reflections, we find $\phi = 76° 50'$, $\phi' = 46° 55'$, $D = 318° 20' = 360° - 41° 40'$, and they therefore emerge from the drop in a direction not differing greatly from that of the direct light of the sun, and are quite invisible.

The corresponding numbers for four reflections are $\phi = 79° 38'$, $\phi' = 47° 32'$, $D = 403° 56' = 360° + 43° 56'$; and this bow will be invisible for the same reason.

For five reflections the corresponding numbers are $\phi = 81° 26'$, $\phi' = 48° 52' 10''$, $D = 486° 28' = 540° - 53° 32'$; and this bow will be just outside the second. It may be seen in waterfalls, where the drops are near the eye.

In the laboratory, by the use of a single origin of light and by observing with a telescope, the first nineteen bows have been observed.

220. Supernumerary Bows.

It was stated above that the explanation of the rainbow afforded by the method of minimum deviation was unsatisfactory. This is shewn by the phenomenon, occasionally observed, of bands of colour within the first bow and outside the second. The colour seen is usually red, though green and violet may follow alternately. They are most noticeable in very fine rain.

Their explanation is due to Airy, who on the principles of Physical Optics shewed that the rainbow must lie slightly within the position given by the geometrical theory, and that there will also be these supernumerary bows within the first. A very clear account is given in Verdet, *Œuvres*, t. IV. pp. 774—791.

A similar phenomenon is observed in the *white bow*, which is a circle of radius between 38° and 41° 5′, seen in fine mist.

For an explanation of halos, parhelia, and other phenomena due to the presence of ice-crystals in the atmosphere, cf. Verdet, *loc. cit.* pp. 793—810.

221. Example. *Prove that for the nth rainbow, the limiting ratio of the radius of curvature of the caustic at any point to the cube of the distance of that point from the raindrop, as the point approaches the minimum deviation position, is*

$$\frac{2}{a^2}\left\{\frac{n(n+2)}{n+1}\right\}^2 \frac{\{(n+1)^2-\mu^2\}^{\frac{1}{2}}}{(\mu^2-1)^{\frac{3}{2}}}.$$

The perpendicular from the centre of the drop on the emergent ray is $a\sin\phi$, and the angle ω which it makes with a fixed straight line is equal to $2\{\phi-(n+1)\phi'\}$.

Hence
$$\frac{d\phi}{d\omega}=\frac{\mu\cos\phi'}{2\{\mu\cos\phi'-(n+1)\cos\phi\}},$$

and
$$\frac{d^2\phi}{d\omega^2}=\frac{d\phi}{d\omega}\frac{d}{d\phi}\frac{\mu\cos\phi'}{2\{\mu\cos\phi'-(n+1)\cos\phi\}}$$

$$=-\frac{(n+1)(\mu^2-1)\sin\phi}{4\{\mu\cos\phi'-(n+1)\cos\phi\}^3}.$$

Also

$$\rho\equiv p+\frac{d^2p}{d\omega^2}=a\sin\phi+a\cos\phi\frac{d^2\phi}{d\omega^2}-a\sin\phi\left(\frac{d\phi}{d\omega}\right)^2$$

$$=a\sin\phi\left[1-\frac{(n+1)(\mu^2-1)\cos\phi}{4\{\mu\cos\phi'-(n+1)\cos\phi\}^3}-\frac{\mu^2\cos^2\phi'}{4\{\mu\cos\phi'-(n+1)\cos\phi\}^2}\right].$$

The distance v of the primary focus behind the point of emergence is $\frac{1}{2}a\cos\phi\dfrac{(2n+2)\cos\phi-\mu\cos\phi'}{(n+1)\cos\phi-\mu\cos\phi'}$ (v, Art. 214), while for the ray of minimum deviation $\mu\cos\phi'-(n+1)\cos\phi=0$.

Hence for that ray the limiting value of $\frac{\rho}{v^3}$ is

$$\frac{2}{a^2}\frac{(\mu^2-1)\sin\phi}{(n+1)^2\cos^6\phi},$$

or
$$\frac{2}{a^2}\left\{\frac{n(n+2)}{n+1}\right\}^2\frac{\{(n+1)^2-\mu^2\}^{\frac{1}{2}}}{(\mu^2-1)^{\frac{3}{2}}},$$

on substituting the values of $\sin\phi$ and $\cos\phi$.

Further, at the point on the caustic where the ray crosses within the circle, $(2n+2)\cos\phi=\mu\cos\phi'$, and at this point

$$\rho=-\frac{a}{4}\frac{(\mu^2-1)\sin\phi}{(n+1)^2\cos^2\phi}=-\frac{a}{4}\frac{\{(2n+2)^2-\mu^2\}^{\frac{1}{2}}\{(2n+1)(2n+3)\}^{\frac{1}{2}}}{(n+1)^2}.$$

For the final point on the caustic $\phi=\frac{1}{2}\pi$, and $\rho=\frac{3}{4}a$, for all values of n.

This change of sign in ρ proves that there is always a cusp within the circle between these two latter points.

EXAMPLES.

1. Shew that if $T=0$ and $N=0$ are the equations of the tangent and normal at any point of a reflecting curve, the equation of the reflected ray is $T/T'+N/N'=0$, where T', N' are what T, N become when the coordinates of the bright point are substituted.

The caustic of an ellipse is as a rule of the sixth class; the caustic of a parabola is of the fifth class, unless the vertex be the origin of light, when it is of the fourth class.

2. Rays are incident parallel to the axis of y on a reflecting curve; shew that the caustic is given by eliminating x and y between the equation of the curve and the equations

$$(x-\xi)\frac{d^2y}{dx^2}=\frac{dy}{dx}, \quad (\eta-y)\frac{d^2y}{dx^2}=\tfrac{1}{2}\left\{1-\left(\frac{dy}{dx}\right)^2\right\}.$$

The reflecting curve is $x/c=\log(y/c)$; shew that the caustic is the catenary

$$\eta=c\cosh\{(\xi+c)/c\}.$$

3. Rays from the focus of a parabola are reflected at the evolute of the parabola; shew that the caustic is the evolute of a parabola.

4. If rays from a luminous point be reflected at a parabola, the caustic has three asymptotes, reflected from points at a finite distance, except when the luminous point is on the axis, when there are only two.

5. A sun-dial is constructed of a reflecting cylinder, whose cross-section is a cycloid mounted on a card so that the generating lines of the cylinder are parallel to the earth's axis and perpendicular to the plane of the card, while the axis of the cycloidal cross-section lies in the plane of the meridian. Prove that the distance from the axis of the further point in which the caustic due to the reflection of the sun's rays cuts the line joining the cusps of the cycloid will always indicate apparent solar time.

The caustic by reflection at a cycloid for parallel rays is the envelope of a straight line rigidly attached to and moving with the generating circle of the cycloid.

6. The equation of the caustic formed by reflection at a circle of parallel rays is

$$\left(\frac{2r}{a}\right)^{\frac{2}{3}}=\cos^{\frac{2}{3}}\frac{\theta}{2}+\sin^{\frac{2}{3}}\frac{\theta}{2};$$

the centre of the circle being the origin.

7. Shew that the relation between the coordinates of the perpendicular from the centre on the tangent to the caustic by reflection at a circle is

$$2\cos^{-1}(p/a)+\cos^{-1}(p/c)=\omega,$$

where a is the radius of the circle, c the distance of the radiant point from the centre.

Hence shew that for cusps on the caustic p is 0 or c.

8. Any curve whose equation in polar coordinates is $f(\theta, \phi)=0$ is drawn on the inside of a polished reflecting sphere of radius a; shew that the image, formed by primary foci, which is seen by an eye at the pole is the intersection of a sphere of radius $\frac{(n+1)a}{2n+1}$ through the eye with a cone whose equation is

$$f\{(2n+2)\theta, \phi\}=0.$$

9. Prove that, if the origin be a radiant point, those points on a known curve which give rise to asymptotes on the caustic by reflection are given by the equation $\frac{d}{d\varpi}\left(\frac{\varpi}{r}\right)=0$, and those which give rise to cusps by the equation $\frac{d^2}{d\varpi^2}\left(\frac{\varpi}{r}\right)=0$; where $\varpi=p^2$, and $f(r, p)=0$ is the pedal equation of the curve.

10. A pencil of rays parallel to the axis of y is reflected at a curve; shew that the arc of the caustic between any two points is the difference of the corresponding values of $\operatorname{cosec} 2\phi \frac{d}{d\phi}(y\cos^2\phi)$, where ϕ is the angle of incidence.

The arc of the caustic for parallel rays reflected at the curve $y=a\log(\sec x/a)$ is the difference of the ordinates at the points of incidence.

11. Rays from the origin are reflected at any plane curve. Let the path U from the origin to any point (x, y) on a reflected ray be expressed in terms of (x, y) and the initial angle of divergence θ of the ray from a fixed axis. Shew that the caustic is obtained by eliminating θ from the equations $\frac{\partial U}{\partial\theta}=0$ and $\frac{\partial^2 U}{\partial\theta^2}=0$; and that the radius of curvature of the caustic at any point is $\frac{v^3}{u^3}\frac{\partial^3 U}{\partial\theta^3}$; where u and v are the distances from the point of reflection of the origin and the primary focus respectively.

12. A pencil of rays diverging from a point is refracted at any plane curves. If Q' be a point on the caustic, P the last point of refraction corresponding to Q', and ϕ' the last angle of refraction, shew that the radius of curvature of the caustic at Q' is $v\tan\phi'\frac{dV'}{dV}$; where V and V' are the reduced paths of the ray up to the points P and Q' respectively, expressed in terms of a parameter defining the ray, and v is the distance PQ'.

13. Rays from the origin of coordinates are refracted from a medium of index μ into one of index μ' at a curve, whose equation is given in terms of r and p. Shew that the equation in similar coordinates r' and p' of a certain orthotomic curve of the refracted rays can be found from the equations

$$\mu^2 r'^2 - 2\mu\mu' r' p - (\mu'^2 - \mu^2) r^2 = 0,$$
$$p'^2/r'^2 + p^2/r^2 = 1.$$

14. Shew that to the eye at height h above the surface of a pond whose bottom is flat and depth is d, the bottom as seen by primary foci appears a surface of revolution generated by the curve $(x^2\cos^2 a + y^2)^3 (y-h)^2 = y^6 d^2 \sin^2 a$ rotating about the axis of y; where the axes are taken horizontal and vertical through the eye, and a is the critical angle.

15. Rays diverging from a point on the axis of x are refracted at a circle of radius a; shew that the coordinates of any point on the caustic are given by the equations

$$x = -a \sin\phi' \cos(\phi - \phi' - \theta - \omega) \operatorname{cosec}\omega,$$

$$y = -a \sin\phi' \sin(\phi - \phi' - \theta - \omega) \operatorname{cosec}\omega,$$

where $$\tan\omega = \tan\phi - \tan\phi' - \tan\theta,$$

ϕ and ϕ' being the angles of incidence and refraction, and θ the angle of divergence of the ray from the axis of x.

Shew that if the distance of the bright point from the centre of the circle lie between a and $\mu a/(\mu - 1)$, the caustic for rays refracted at the convex part of the circle has two asymptotes.

16. Rays from a point on a circle of radius a are refracted at the circle into a medium of refractive index $\mu\,(>1)$. Shew that the entire length of the caustic is

$$2a\left\{\frac{(\mu^2-1)^{\frac{3}{2}}}{\mu} - \frac{2(\mu-1)^2}{(2\mu-1)\mu}\right\};$$

and that the radius of curvature of the caustic at any point is

$$6a \sin\phi \cos\phi' (\mu\cos\phi' - \cos\phi)^2/(2\mu\cos\phi' - \cos\phi)^3.$$

17. A pencil of rays parallel to the axis of x is refracted at a circle of radius a; shew that an orthotomic curve of the refracted rays is given by the equations

$$x = -a\cos\phi\cos\phi' \sin(\phi - \phi') \operatorname{cosec}\phi,$$

$$y = a\{1 - \cos\phi\cos\phi'\cos(\phi - \phi')\} \operatorname{cosec}\phi.$$

Shew that the radius of curvature of the caustic of the refracted rays at any point is equal to

$$3a \sin\phi\cos\phi \sin^2\phi' \cos\phi' \operatorname{cosec}^2(\phi - \phi');$$

and that the length of the caustic from cusp to cusp is

$$a\{(\mu^3+1) - (\mu^2-1)^{\frac{3}{2}}\}/\mu(\mu^2-1).$$

18. A small pencil of rays diverging from a point is refracted at a sphere of radius a and emerges after n internal reflections. Prove that if U be the distance of the origin of light from the middle point of the chord of the sphere intercepted on the incident ray, and V the distance of the primary focus from the middle point of the chord intercepted on the emergent ray,

$$\frac{1}{U} + \frac{1}{V} = \frac{2}{a}\left(\frac{1}{\cos\phi} - \frac{n+1}{\mu\cos\phi'}\right).$$

19. A small beam of parallel rays is incident at angle ϕ on a refracting sphere of radius a. Shew that after passing through the sphere, the focal lines are at distances from the sphere measured along the beam equal respectively to

$$\tfrac{1}{2}a\cos\phi\,(2\cos\phi\sin\phi' - \cos\phi'\sin\phi) \operatorname{cosec}(\phi - \phi'),$$

and $$a\sin(2\phi' - \phi) \operatorname{cosec} 2(\phi - \phi').$$

20. Prove that when parallel rays are incident on a refracting sphere the primary focus of a small pencil divides the ray between the rth and $(r+1)$th internal reflections in the ratio

$$\mu \cos \phi' - 2r \cos \phi \ : \ \mu \cos \phi' - (2r+2) \cos \phi'.$$

Shew that the distance of the secondary focus after the rth reflection from the point of reflection is $a \sin (\phi - 2r\phi')/\sin \{\phi - (2r+1)\phi'\}$, where a is the radius of the sphere.

21. A small circular cylinder of rays is incident on a refracting sphere of radius a at angle ϕ, so that after n internal reflections it emerges with minimum deviation. Shew that the rays pass through two equal circles, at distances along the axis from the point of emergence equal to

$$a \sin (n+1)\phi' \operatorname{cosec}\{(n+1)\phi' - \phi\}$$

and $a \cos (n+1)\phi' \sec \{(n+1)\phi' - \phi\}$, where $\mu \cos \phi' = (n+1) \cos \phi$.

22. Assuming the angular diameter of the sun to be 30′, shew that in the primary bow the colours, whose refractive indices are μ and $\mu + \delta\mu$ will be separated only if

$$\delta\mu > \frac{\pi\mu}{720} \left(\frac{\mu^2 - 1}{4 - \mu^2}\right)^{\frac{1}{2}}.$$

23. A rainbow is seen by reflection at the surface of still water by an eye at height h above the water. Shew that the bow on the surface of the water is a hyperbola, of eccentricity $\cos a/\cos \theta$, with vertex at a horizontal distance $h \cot (\theta - a)$ from the eye, and asymptotes parallel to the straight lines from the eye to the points of intersection of a horizontal plane through the eye with the corresponding bow seen directly; where a is the altitude of the sun, and θ the supplement of the minimum deviation in the bow seen directly.

CHAPTER XIII.

HETEROGENEOUS MEDIA.

222. WHEN the density and refractive index of a transparent medium vary continuously from point to point, the course of a ray of light through it will be a curve, plane or twisted, of continuous curvature. The atmosphere is an instance of such a medium, and the apparent position of a star, determined by the tangent to the ray as it enters the eye, therefore differs from the true position in the heavens.

In the general case, when the index of refraction is a known function of the coordinates, we can obtain a differential equation of the second order for the course of a ray; the shape of a small pencil of rays must then be found by calculating the reduced path for the rays of the pencil, or by solution of the differential equation of the characteristic function.

Before considering the general case we shall determine the course of a ray directly when the medium is stratified (i) in parallel planes, (ii) in concentric spheres.

223. Medium stratified in parallel planes.

Let the surfaces of equal refractive index be planes parallel to that of yz, so that μ is a known function of x. The path of any ray will lie throughout in its first plane of incidence, since that plane contains all the normals to the successive refracting surfaces. Taking this plane as the plane of xy, we have, if ϕ be the angle of incidence at any point where the refractive index is μ,

$$\mu \sin \phi = \mu_0 \sin \phi_0.$$

Hence $\mu^2 = \mu_0^2 \sin^2 \phi_0 \operatorname{cosec}^2 \phi = \mu_0^2 \sin^2 \phi_0 \left\{1 + \left(\dfrac{dx}{dy}\right)^2\right\}$, and the path of any ray in this plane is therefore given by the equation

$$y = \mu_0 \sin \phi_0 \int^x (\mu^2 - \mu_0^2 \sin^2 \phi_0)^{-\frac{1}{2}} \, dx.$$

The characteristic function, which determines the orthotomic surfaces of a small pencil, has been determined above (cf. Art. 186).

224. Medium stratified in concentric spheres.

If the refracting surfaces be concentric spheres, so that μ is a function of the distance from the centre, the path of any ray will lie throughout in one plane, since the plane of incidence and refraction, which is always the osculating plane of the path, necessarily contains a fixed point, and is therefore fixed.

Taking the plane of the ray as the plane of reference, we have at any refraction $\mu \sin\phi = \mu' \sin\phi'$, and $\sin\phi = p/r$, $\sin\phi' = p'/r$, where p and p' are the perpendiculars from the centre on the incident and refracted rays.

Hence $\mu p = \mu' p'$; and the path of any ray is given by the equation $\mu p = h$, or if $u = 1/r$, by the equation

$$\mu^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}.$$

Assuming $\dfrac{du}{d\theta}$ positive, *i.e.* that the ray is approaching the centre, we obtain for the path of the ray the equation

$$\theta = \int^u \frac{h\,du}{(\mu^2 - h^2u^2)^{\frac{1}{2}}}.$$

225. Orthotomic surfaces of a small pencil.

If a small pencil diverging from a point be refracted at concentric spheres, the orthotomic surfaces will be surfaces of revolution about the line joining the origin of light to the centre, and the secondary focus of the pencil will lie throughout on the axis of revolution.

Let V be the reduced path from the point $r = a$ to any point on the ray, then

$$\left.\begin{aligned}\frac{\partial V}{\partial r} &= \mu\frac{dr}{ds} = -\mu\cos\phi = -(\mu^2 - h^2u^2)^{\frac{1}{2}}\\ \frac{\partial V}{r\partial\theta} &= \mu r\frac{d\theta}{ds} = \mu\sin\phi = hu\end{aligned}\right\}\quad\text{.....................(i)},$$

it being supposed that the ray is approaching the centre and is concave to that point.

Hence
$$V = h\theta - \int_a^r (\mu^2 - h^2u^2)^{\frac{1}{2}}\,dr$$

$$= \int_{1/a}^u \frac{\mu^2\,du}{u^2(\mu^2 - h^2u^2)^{\frac{1}{2}}}\quad\text{..............................(ii)},$$

on substituting for θ from the equation giving the path of the ray.

For the equation of the meridian curve of the orthotomic surface determined by equating V to a constant, the parameter h must be eliminated between the equations giving V and θ in terms of r.

If however we retain h as a function of r and θ, we obtain, on differentiating the equation for the path,

$$1=\frac{\partial h}{\partial \theta}\int_{1/a}^{u} \mu^2 (\mu^2-h^2u^2)^{-\frac{3}{2}} du,$$

$$0=\frac{\partial h}{\partial r}\int_{1/a}^{u} \mu^2 (\mu^2-h^2u^2)^{-\frac{3}{2}} du - hu^2 (\mu^2-h^2u^2)^{-\frac{1}{2}};$$

and on differentiating equations (i)

$$\frac{\partial^2 V}{\partial r^2} = -(\mu^2-h^2u^2)^{-\frac{1}{2}}\left\{\mu\frac{d\mu}{dr}+h^2u^3-hu^2\frac{\partial h}{\partial r}\right\},$$

$$\frac{\partial}{\partial r}\left(\frac{\partial V}{r\partial \theta}\right) = -hu^2+u\frac{\partial h}{\partial r}, \quad \frac{\partial^2 V}{\partial \theta^2}=\frac{\partial h}{\partial \theta}.$$

Hence, on substituting in the formula

$$\left\{\left(\frac{\partial V}{\partial r}\right)^2+\left(\frac{\partial V}{r\partial \theta}\right)^2\right\}\Big/\rho=\frac{\partial^2 V}{\partial r^2}\left(\frac{\partial V}{r\partial \theta}\right)^2-2\frac{\partial}{\partial r}\left(\frac{\partial V}{r\partial \theta}\right)\frac{\partial V}{\partial r}\frac{\partial V}{r\partial \theta}$$
$$+\left(\frac{\partial^2 V}{r^2\partial \theta^2}+\frac{\partial V}{r\partial r}\right)\left(\frac{\partial V}{\partial r}\right)^2,$$

the distance of the primary focus from the point of incidence is given by the equation

$$\frac{\mu^3}{\rho}=\frac{\mu^4u^2}{\mu^2-h^2u^2}\left[\int_{1/a}^{u}\frac{\mu^2 du}{(\mu^2-h^2u^2)^{\frac{3}{2}}}\right]^{-1}-\frac{\mu^4u+h^2u^2\mu\dfrac{d\mu}{dr}}{(\mu^2-h^2u^2)^{\frac{1}{2}}}.$$

This value for the distance of the primary focus will not be the same as that found by successive applications of the formula of Art. 176; the orthotomic surfaces considered here are drawn in the heterogeneous medium, paying attention to the change in the refractive index at each point of the surfaces; the orthotomic surfaces considered in Chapter XII. were drawn each in succession entirely in homogeneous media.

226. General equations of the path of a ray.

Let the refractive index μ be a known function of the coordinates (x, y, z), and let a ray be refracted from the medium of index μ into the medium of index $\mu+\partial\mu$ at their surface of separation.

The normal to this surface lies in the plane containing the incident and refracted rays, *i.e.* in the osculating plane of the path. Hence the deviation $(\phi-\phi')=$ the angle of contingence $\partial\epsilon$, and by the law of refraction $\mu\sin\phi=(\mu+\partial\mu)\sin(\phi-\partial\epsilon)$,

or $$\partial\epsilon=\frac{\partial\mu}{\mu}\tan\phi. \qquad \text{(i)}.$$

But if $PQ\,(\equiv \partial s)$ be an element of the path, and if $PN\,(\equiv \partial n)$ be the intercept made on the *principal normal* to the path between the surfaces μ and $\mu + \partial\mu$,

$$\tan\phi = PQ/PN,$$

in the limit.

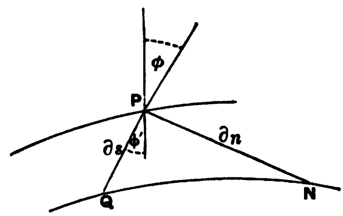


Fig. 104.

Hence $$\frac{\mu}{\rho} = \frac{\partial\mu}{\partial n} \quad\text{..........(ii).}$$

In Cartesian coordinates, the direction-cosines of the principal normal being

$$\rho\frac{d^2x}{ds^2},\quad \rho\frac{d^2y}{ds^2},\quad \rho\frac{d^2z}{ds^2},$$

this equation is

$$\frac{\mu}{\rho} = \rho\left\{\frac{d^2x}{ds^2}\frac{\partial\mu}{\partial x} + \frac{d^2y}{ds^2}\frac{\partial\mu}{\partial y} + \frac{d^2z}{ds^2}\frac{\partial\mu}{\partial z}\right\} \quad\text{..............(iii);}$$

and the path is theoretically determined by expressing x, y, z in terms of s from this equation and the two equations

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1 \quad\text{.................(iv),}$$

$$\begin{vmatrix} \dfrac{\partial\mu}{\partial x}, & \dfrac{\partial\mu}{\partial y}, & \dfrac{\partial\mu}{\partial z} \\[2ex] \dfrac{dx}{ds}, & \dfrac{dy}{ds}, & \dfrac{dz}{ds} \\[2ex] \dfrac{d^2x}{ds^2}, & \dfrac{d^2y}{ds^2}, & \dfrac{d^2z}{ds^2} \end{vmatrix} = 0 \quad\text{....................(v),}$$

the latter equation expressing that the osculating plane of the path contains the normal to the refracting interface.

If the medium be symmetrical on either side of the plane of xy and the ray be incident in that plane, its path will lie entirely in that plane; and equation (ii) determining the path may be written as

$$\frac{\mu}{\rho} = \frac{dx}{dn}\frac{\partial\mu}{\partial x} + \frac{dy}{dn}\frac{\partial\mu}{\partial y} = \begin{vmatrix} \dfrac{dx}{ds}, & \dfrac{dy}{ds} \\[2ex] \dfrac{\partial\mu}{\partial x}, & \dfrac{\partial\mu}{\partial y} \end{vmatrix};$$

or, if $p = \dfrac{dy}{dx}$, the differential equation for the path is

$$\frac{\dfrac{dp}{dx}}{1+p^2} = \frac{\dfrac{\partial\mu}{\partial y} - p\dfrac{\partial\mu}{\partial x}}{\mu} \quad\text{.......................(vi).}$$

When μ is given in terms of the polar coordinates $r(\equiv 1/u)$ and θ, the differential equation for the path is

$$\frac{\mu}{\rho} = \begin{vmatrix} \dfrac{dr}{ds}, & \dfrac{rd\theta}{ds} \\ \dfrac{\partial\mu}{\partial r}, & \dfrac{\partial\mu}{r\partial\theta} \end{vmatrix}$$

or

$$\frac{u + \dfrac{d^2u}{d\theta^2}}{u^2 + \left(\dfrac{du}{d\theta}\right)^2} = \frac{\dfrac{\partial\mu}{\partial u} - \dfrac{du}{u^2 d\theta}\dfrac{\partial\mu}{\partial\theta}}{\mu} \quad\text{.................(vii).}$$

When the path of the ray is given, equations (vi) and (vii) are partial differential equations for the possible forms of the refractive index to be solved by Lagrange's rule.

227. Newton's Theorem.

The path of a ray in any medium is that of a particle moving freely with velocity μ under the action of a conservative system of forces.

The equations of motion of a particle of unit mass moving with velocity v under the action of a system of forces, whose potential is V, are

$$v\frac{dv}{ds} = -\frac{\partial V}{\partial s}, \quad \frac{v^2}{\rho} = -\frac{\partial V}{\partial n}, \quad 0 = \frac{\partial V}{\partial \nu} \quad\text{..............(i),}$$

where ∂s, ∂n, $\partial \nu$ are infinitesimal displacements along the tangent, principal normal, and binormal of the path.

If $v = \mu$, and $V = E - \frac{1}{2}\mu^2$, the first of these equations is satisfied identically, while the second and third are equations (ii) and (v) of the preceding article. Hence we may re-write the equations for the path in the form

$$\left.\begin{aligned} \mu\frac{d}{ds}\left(\mu\frac{dx}{ds}\right) &= \mu\frac{\partial\mu}{\partial x} \\ \mu\frac{d}{ds}\left(\mu\frac{dy}{ds}\right) &= \mu\frac{\partial\mu}{\partial y} \\ \mu\frac{d}{ds}\left(\mu\frac{dz}{ds}\right) &= \mu\frac{\partial\mu}{\partial z} \end{aligned}\right\} \quad\text{...................(ii),}$$

or in any other form suggested by the methods of Dynamics.

This latter form of the equations for the path may also be deduced directly from the equations of Article 19, connecting the direction-cosines of the incident and refracted rays. With the notation of that article, $\mu' l' - \mu l$ is proportional to p, where (p, q, r) are the direction-cosines of the normal to the refracting interface.

In the case of a path of continuous curvature, this equation is

$$\frac{d}{ds}\left(\mu \frac{dx}{ds}\right) = R \frac{\partial \mu}{\partial x},$$

that is,
$$\frac{\partial \mu}{\partial s}\frac{dx}{ds} + \mu \frac{d^2 x}{ds^2} = R \frac{\partial \mu}{\partial x}.$$

On multiplying this and the two similar equations in y and z by

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds},$$

and adding, we obtain unity for the value of the multiplier R, since

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0.$$

228. Conjugate functions.

If α, β are conjugate functions of the rectangular coordinates x, y, the two-dimensional path of a ray is given by the equations

$$h\frac{d}{ds}\left(\mu h^2 \frac{d\alpha}{ds}\right) = \frac{\partial}{\partial \alpha}(\mu h),$$

$$h\frac{d}{ds}\left(\mu h^2 \frac{d\beta}{ds}\right) = \frac{\partial}{\partial \beta}(\mu h),$$

where h is the modulus $\left|\dfrac{d(x+iy)}{d(\alpha+i\beta)}\right|$.

If a functional relation exist between the complex quantities $x+iy$ and $\alpha+i\beta$, it is well known that the curves $\alpha = \text{constant}$, $\beta = \text{constant}$, cut at right angles; and that if $x + iy = F(\alpha + i\beta)$, then

$$d(x+iy) = F'(\alpha+i\beta)\, d(\alpha+i\beta),$$

whence
$$ds = |F'(\alpha+i\beta)|\, \{(d\alpha)^2 + (d\beta)^2\}^{\frac{1}{2}}$$
$$= h\, \{(d\alpha)^2 + (d\beta)^2\}^{\frac{1}{2}},$$

where
$$h = |F'(\alpha+i\beta)| = \left\{\frac{\partial(x, y)}{\partial(\alpha, \beta)}\right\}^{\frac{1}{2}} = \left\{\frac{\partial(\alpha, \beta)}{\partial(x, y)}\right\}^{-\frac{1}{2}}.$$

Consider a unit particle moving with velocity μ; then with the notation of Dynamics we have

$$2T = \mu^2 = h^2(\dot{\alpha}^2 + \dot{\beta}^2),$$

where the time-variation is introduced by the equation $\mu = \dfrac{ds}{dt}$.

Applying Lagrange's method to form the equations of motion of the particle under forces of potential $-\tfrac{1}{2}\mu^2$, we obtain the equation

$$\frac{d}{dt}(h^2\dot{\alpha}) - (\dot{\alpha}^2 + \dot{\beta}^2)\, h\frac{\partial h}{\partial \alpha} = \mu\frac{\partial \mu}{\partial \alpha},$$

and a similar equation in β.

The former equation may be written as

$$\mu\frac{d}{ds}\left(\mu h^2\frac{d\alpha}{ds}\right) = \frac{\mu^2}{h}\frac{\partial h}{\partial \alpha} + \mu\frac{\partial \mu}{\partial \alpha},$$

that is,
$$h\frac{d}{ds}\left(\mu h^2\frac{d\alpha}{ds}\right) = \frac{\partial}{\partial \alpha}(\mu h);$$

similarly
$$h\frac{d}{ds}\left(\mu h^2\frac{d\beta}{ds}\right) = \frac{\partial}{\partial \beta}(\mu h).$$

These equations for the ray may also be obtained by making the reduced path $\int \mu h\,\{(d\alpha)^2 + (d\beta)^2\}^{\frac{1}{2}}$ stationary in value, as in Article 230.

Let ω be the angle between the ray and the curve α = constant; then since the intercepts made on the curves β and α by consecutive orthogonal curves are respectively $hd\alpha$ and $hd\beta$, these equations may be written

$$h\frac{d}{ds}(\mu h \sin\omega) = \frac{\partial}{\partial \alpha}(\mu h),$$

$$h\frac{d}{ds}(\mu h \cos\omega) = \frac{\partial}{\partial \beta}(\mu h).$$

If μh be constant, *i.e.* if the index of refraction vary as the square root of the Jacobian $\{\partial(\alpha, \beta)/\partial(x, y)\}$, these equations give ω = constant. Since the equation of an oblique trajectory of the curves α = constant may be written as $hd\alpha/\sin\omega - hd\beta/\cos\omega = 0$, we see that the path of a ray is given by the equation

$$U = \alpha\cos\omega - \beta\sin\omega = \text{constant},$$

and the curves orthotomic to the rays are given by the equation

$$V = \alpha\sin\omega + \beta\cos\omega = \text{constant}.$$

Hence if $\alpha + i\beta = f(x+iy)$, the rays and their orthotomic curves are included in the relation

$$U + iV = f(x+iy)\exp(i\omega).$$

The reduced path between any two points of a ray defined by U is equal to the difference of the values of V for those points multiplied by the constant quantity μh.

That the path of a ray is an oblique trajectory of the curves of reference, if μh be a constant, is at once obvious from the fact that the reduced path $\int \mu h \{(d\alpha)^2 + (d\beta)^2\}^{\frac{1}{2}}$ is with that condition proportional to $\int \{(d\alpha)^2 + (d\beta)^2\}^{\frac{1}{2}}$, and is therefore a minimum when a linear relation exists between α and β. For this integral represents the distance between two points referred to rectangular Cartesian coordinates α and β, and is a minimum when taken along a straight line.

The forms of μ possible that the path may be an oblique trajectory are necessarily restricted; for since

$$\mu = \left| \frac{d(\alpha + i\beta)}{d(x+iy)} \right| = |f'(x+iy)|,$$

$\log \mu$ is the real part of a function of the complex variable, and therefore satisfies Laplace's equation $\nabla^2 (\log \mu) = 0$.

If for instance μ be a function of x only, the only possible form of $\log \mu$ is $A + Bx$, and the conjugate functions α, β are given by the equation

$$\alpha + i\beta = \mu_0 a \exp\{(x+iy)/a\},$$

whence

$$\mu = \mu_0 \exp(x/a).$$

The paths of the rays and their orthotomic curves are therefore of the form

$$U + iV = \mu_0 a \exp[\{x + i(y + a\omega)\}/a],$$

i.e. they are all catenaries of equal strength having their directrices parallel to the axis of y.

229. Medium stratified cylindrically.

Let the refractive index be a function of the coordinates x and y only; if a ray be incident on the medium in a direction oblique to the axis of z, its path will no longer lie in one plane, but will be a tortuous curve and will be given as the intersection of two surfaces.

From the third of equations (ii) Article 227 we deduce $\mu \dfrac{dz}{ds} = c$; this is equation (i) Article 18.

If ds' be the elementary arc of the projection of the ray on the plane of xy,

$$(ds')^2 = (dx)^2 + (dy)^2 = (ds)^2 - (dz)^2 = \frac{\mu^2 - c^2}{\mu^2}(ds)^2.$$

Hence the equations (ii) for the path of the ray may be written

$$\left.\begin{aligned}\frac{d}{ds'}\left(\sqrt{\mu^2-c^2}\frac{dx}{ds'}\right)&=\frac{\mu}{\sqrt{\mu^2-c^2}}\frac{\partial\mu}{\partial x}\\ \frac{d}{ds'}\left(\sqrt{\mu^2-c^2}\frac{dy}{ds'}\right)&=\frac{\mu}{\sqrt{\mu^2-c^2}}\frac{\partial\mu}{\partial y}\end{aligned}\right\}.$$

It therefore follows that in a medium stratified cylindrically the projection of a ray on a plane perpendicular to the axis is the two-dimensional path of a ray in a medium of index $\sqrt{\mu^2-c^2}$. This may also be at once deduced from equation (iii) Article 18.

Further, if the index of refraction be a function only of the distance from the axis of z, we may with advantage use the cylindrical coordinates r, ϕ and z. Putting

$$u=\mu\frac{dr}{ds},\quad v=\mu r\frac{d\phi}{ds},\quad w=\mu\frac{dz}{ds},$$

and making use of the equations of motion of a particle in cylindrical coordinates, the general equations for the path are, since $\dot{\phi}=\frac{d\phi}{dt}=\mu\frac{d\phi}{ds}$,

$$\left.\begin{aligned}\mu\left(\frac{du}{ds}-v\frac{d\phi}{ds}\right)&=\mu\frac{\partial\mu}{\partial r}\\ \mu\left(\frac{dv}{ds}+u\frac{d\phi}{ds}\right)&=\mu\frac{\partial\mu}{r\partial\phi}\\ \mu\frac{dw}{ds}\qquad\qquad&=\mu\frac{\partial\mu}{\partial z}\end{aligned}\right\}.$$

These equations are equivalent to only two independent equations, as they reduce to an identity when we multiply by u, v, w, and add. If μ be a function of r only, the second and third are integrable, and give

$$\mu r^2\frac{d\phi}{ds}=h,\quad \mu\frac{dz}{ds}=c.$$

Hence $$\left(\frac{dr}{ds}\right)^2=1-\frac{h^2}{\mu^2r^2}-\frac{c^2}{\mu^2}=\frac{(\mu^2-c^2)r^2-h^2}{\mu^2r^2},$$

and therefore the path of a ray is found by integrating the equations

$$\frac{dz}{dr}=\frac{cr}{\{(\mu^2-c^2)r^2-h^2\}^{\frac{1}{2}}},$$

$$r\frac{d\phi}{dr}=\frac{h}{\{(\mu^2-c^2)r^2-h^2\}^{\frac{1}{2}}}.$$

This curve is the intersection of a surface of revolution and a cylinder.

230. Characteristic Function.

The path of any ray in a heterogeneous medium may be found by making the reduced path $\int \mu ds$, taken between two points A and B on the ray, stationary in value. If $V = \int_A^B \mu ds$ and δ denote small arbitrary variation in the form of the path,

$$\delta V = \int_A^B (\delta\mu ds + \mu \delta ds).$$

But $$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2,$$

and therefore $$\begin{aligned} ds\,(\delta ds) &= dx\,(\delta dx) + dy\,(\delta dy) + dz\,(\delta dz) \\ &= dx\,(d\delta x) + dy\,(d\delta y) + dz\,(d\delta z), \end{aligned}$$

since the order of the symbols d and δ is interchangeable. Hence

$$\begin{aligned} \int_A^B \mu\delta ds &= \int_A^B \left(\mu \frac{dx}{ds}\, d\delta x + \mu \frac{dy}{ds}\, d\delta y + \mu \frac{dz}{ds}\, d\delta z\right) \\ &= \left[\mu \frac{dx}{ds}\,\delta x + \mu \frac{dy}{ds}\,\delta y + \mu \frac{dz}{ds}\,\delta z\right]_A^B \\ &\quad - \int_A^B \left\{\delta x \frac{d}{ds}\left(\mu \frac{dx}{ds}\right) + \delta y \frac{d}{ds}\left(\mu \frac{dy}{ds}\right) + \delta z \frac{d}{ds}\left(\mu \frac{dz}{ds}\right)\right\} ds; \end{aligned}$$

and $$\begin{aligned} \delta V = &\left[\mu \frac{dx}{ds}\,\delta x + \mu \frac{dy}{ds}\,\delta y + \mu \frac{dz}{ds}\,\delta z\right]_A^B \\ &+ \int_A^B \left[\delta x \left\{\frac{\partial \mu}{\partial x} - \frac{d}{ds}\left(\mu \frac{dx}{ds}\right)\right\} + \delta y \left\{\frac{\partial \mu}{\partial y} - \frac{d}{ds}\left(\mu \frac{dy}{ds}\right)\right\} \right. \\ &\qquad\qquad \left. + \delta z \left\{\frac{\partial \mu}{\partial z} - \frac{d}{ds}\left(\mu \frac{dz}{ds}\right)\right\}\right] ds. \end{aligned}$$

Since the variations δx, δy, δz are arbitrary and independent, the path of the ray is given by the equations

$$\left.\begin{aligned} \frac{d}{ds}\left(\mu \frac{dx}{ds}\right) &= \frac{\partial \mu}{\partial x} \\ \frac{d}{ds}\left(\mu \frac{dy}{ds}\right) &= \frac{\partial \mu}{\partial y} \\ \frac{d}{ds}\left(\mu \frac{dz}{ds}\right) &= \frac{\partial \mu}{\partial z} \end{aligned}\right\}.$$

The integrated terms give as conditions at the ends of the path

$$\frac{\partial V}{\partial x} = \mu \frac{dx}{ds}, \quad \frac{\partial V}{\partial y} = \mu \frac{dy}{ds}, \quad \frac{\partial V}{\partial z} = \mu \frac{dz}{ds} \text{ at } B,$$

and $$\frac{\partial V}{\partial x} = -\mu \frac{dx}{ds}, \quad \frac{\partial V}{\partial y} = -\mu \frac{dy}{ds}, \quad \frac{\partial V}{\partial z} = -\mu \frac{dz}{ds} \text{ at } A.$$

These equations (cf. Art. 182) express that the ray is orthogonal at A and B to the surfaces $V = \text{constant}$.

231. Mirage.

A single origin of light sending rays through a heterogeneous medium to an observer will generally appear to be displaced. For the rays pursue curved paths, and the small pencil that enters the eye will diverge from two focal lines on the tangent to the axial ray at the eye. If these lines be not far removed from each other, and be at a considerable distance from the eye, the appearance is that of a single origin of light in the direction of the tangent to the axial ray. Similarly a small object will appear displaced, and moreover, since the angles between the axial rays entering the eye from different points of the object may depend partly on the azimuths of the planes in which they lie, it may appear distorted. Thus the sun, when near the horizon, is displaced by the refraction of the atmosphere and also appears elliptical in shape.

In certain cases the object may appear inverted, but this cannot occur unless the axial rays cross before they reach the eye. In other words, if we retrace the paths of the rays, the rays from the nodal centre of the eye must have an envelope, and must touch that envelope before they reach the object. To obtain an envelope the rays must have been internally reflected at some of the strata of the refracting medium.

In the atmosphere the variations in density and refractive index are not as a rule large enough to produce these displaced and distorted images of terrestrial objects; but if the changes in density are considerable, they give rise to appearances known as the *mirage*.

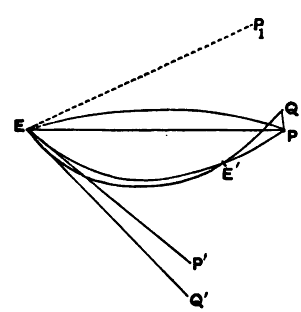


Fig. 105.

In the Arctic regions the temperature of the air sometimes increases rapidly for a certain distance above the sea-level; its density and refractive index will then decrease, and if the eye and an object be in the layer of maximum density, it may be seen at once directly at P, and also at P_1 by rays which pursue a curved path, concave to the earth's surface. For these rays the decrease in the factor μ of the reduced path $\int \mu ds$ may more than counterbalance the increase in the length of the path. As explained above, the apparent

image may be inverted or erect according to the distance of the object. It is stated that on many occasions objects well below the visible horizon have been observed in this manner.

The mirage of the desert is produced by the same cause; the temperature of the lower strata of the air near the heated sand is much above that of the higher strata; the rays may then pursue curved paths convex to the earth's surface, and if the axial rays cross each other, the object may be seen inverted at $P'Q'$, and as it were reflected at the surface of a sheet of water.

232. *To determine the envelope of rays diverging from a point in a heterogeneous medium, and the actual linear magnification produced in a small object.*

Let the index of refraction μ be a function of the coordinate x only; the path of a ray in the plane of xy, starting from a point (x_0, y_0) is given by the equation

$$y - y_0 = \int_{x_0}^{x} \frac{\mu_0 \sin \phi_0}{(\mu^2 - \mu_0^2 \sin^2 \phi_0)^{\frac{1}{2}}} dx \quad \text{......................(i)},$$

where ϕ_0 is the initial angle of incidence (Art. 223).

Hence, differentiating with regard to ϕ_0 to find the envelope of rays diverging from this point, we obtain

$$\int_{x_0}^{x} \frac{\mu^2 dx}{(\mu^2 - \mu_0^2 \sin^2 \phi_0)^{\frac{3}{2}}} = 0 \quad \text{..........................(ii)}.$$

Consecutive rays from the origin of light therefore do not again intersect unless the element of integration in (ii) changes sign, *i.e.* the ray must be incident tangentially at one of the refracting surfaces.

To determine the image of a small object formed by rays passing in the plane of xy, let x_0, y_0 and ϕ_0 vary; μ_0 will vary with x_0. Differentiating the two equations above we obtain

$$\partial y - \partial y_0 = \frac{\mu_0 \sin \phi_0}{\mu \cos \phi} \partial x - \tan \phi_0 \partial x_0 \quad \text{....................(iii)},$$

and

$$\frac{\partial x}{\mu \cos^3 \phi} - \frac{\partial x_0}{\mu_0 \cos^3 \phi_0} + \int_{x_0}^{x} \frac{3\mu^2 \mu_0 \sin \phi_0 \, \partial (\mu_0 \sin \phi_0)}{(\mu^2 - \mu_0^2 \sin^2 \phi_0)^{\frac{5}{2}}} dx = 0 \quad \text{......(iv)}.$$

First, let the small object PQ, of length l, lie along the axial ray; then $\partial x_0 = l \cos \phi_0$, $\partial y_0 = l \sin \phi_0$, and $\partial (\mu_0 \sin \phi_0) = 0$. Hence equation (iii) is

$$\partial y - \tan \phi \, \partial x = \partial y_0 - \tan \phi_0 \, \partial x_0 = 0;$$

so that $P'Q'$, of length l', also lies along the axial ray; and equation (iv) is

$$\frac{l'}{\mu \cos^2 \phi} - \frac{l}{\mu_0 \cos^2 \phi_0} = 0,$$

or

$$\frac{l'}{l} = \frac{\mu \cos^2 \phi}{\mu_0 \cos^2 \phi_0} = \frac{\mu_0 \tan^2 \phi_0}{\mu \tan^2 \phi} \quad \text{..........................(v)}.$$

Secondly, let the small object PQ be perpendicular to the axial ray, so that $\partial x_0 = -l \sin \phi_0$, $\partial y_0 = l \cos \phi_0$. Since the variation of ϕ_0 on passing to the

adjacent point Q is arbitrary, we may take it such that $P'Q'$ is also perpendicular to the axial ray, and use (iv) to determine $\partial\phi_0$. Equation (iii) then gives $\partial y_0 - \tan\phi\,\partial x = \sec\phi\,.\,l' = \partial y_0 - \tan\phi_0\,\partial x_0 = \sec\phi\,.\,l$,

or
$$\frac{l'}{l} = \frac{\cos\phi}{\cos\phi_0} = \frac{\mu_0\tan\phi_0}{\mu\tan\phi}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

In both cases rays that leave P in a plane through the normal other than the plane of reference xy will intersect again in a point in the former plane at the same distance as P' from the normal through P. The image therefore of a small line l at P will be an annulus of thickness l' and breadth depending on the divergence of the pencil from the plane xy as it leaves P.

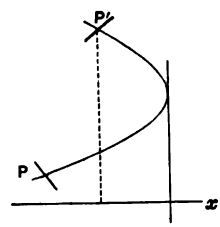


Fig. 106.

In a similar manner we may determine the envelope of rays diverging from a point in a medium stratified in concentric spheres; or we may apply the known solutions of central orbits.

For example it is well known that in elliptic motion about a focus v^2 varies as $(2/r - 1/a)$, and that the envelope of all paths having equal major axes and passing through a given point is a prolate spheroid, the point of contact of any one path with its envelope being at the other end of the chord through the empty focus.

Hence if μ^2 varies as $(1/r - 1/c)$, any small pencil of rays diverging in one plane from P comes to a focus again at P'. An eye E placed between P and P' can therefore either see a small object at P from the front, or it may see its image at P', in which case it will see the back of the object. The apparent directions at E of these images are both along the tangent to the axial path through E, but as the magnification in the primary plane (the plane containing the origin S, P and the axis of the pencil) is different from that in the secondary plane, both apparent images are distorted.

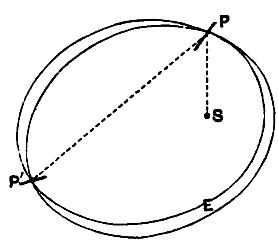


Fig. 107.

EXAMPLES.

1. The refractive index at any point of a medium is $1 + x/a$. Shew that the path of a ray of light incident at the origin at angle α is
$$\cosh\left(\frac{y}{a}\operatorname{cosec}\alpha\right) + \cos\alpha\sinh\left(\frac{y}{a}\operatorname{cosec}\alpha\right) = 1 + \frac{x}{a}.$$

2. A refracting medium is arranged in parallel strata of equal density such that $\mu^2 - 1 = 4ke^{-y/a}$. Shew that the equation of the path of a ray, which is incident at infinity at angle α and passes through the origin, the angle of incidence at that point being β, is
$$e^{y/2a} = \cosh\left(\frac{x}{2a}\cot\alpha\right) + \tan\alpha\cot\beta\sinh\left(\frac{x}{2a}\cot\alpha\right).$$

3. A vessel of depth a, the top and bottom of which are horizontal planes, is filled with a transparent fluid, the refractive index of which at depth z below the surface is $1+z/a$. Two small holes being made in the top, a ray of light enters at one hole, is reflected at the bottom, and emerges at the second hole; shew that the distance between the holes must not exceed

$$2a \log (2+\sqrt{3}).$$

4. A homogeneous medium, of index of refraction cosec a, is bounded by a plane heterogeneous plate, of thickness t, in which the refractive index varies uniformly along the normal from cosec a to unity. Prove that those rays from a point A within the medium, which are incident on the heterogeneous boundary at an angle greater than a, finally become straight lines in the original medium, which intersect the line through A perpendicular to the boundary at points comprised within a length

$$2t \cot\left(\frac{\pi}{4}-\frac{a}{2}\right) \log \cot \frac{a}{2}.$$

5. Shew that if $\mu^2 x$ be a constant, the path of any ray is a cycloid. Find the envelope of all rays through a given focus, and shew that the points on any ray conjugate to a given focus on it are determined by drawing tangents to the ray from the cusp of the cycloid nearest to the focus.

6. A luminous point is placed at the origin of coordinates in a medium for which $\mu=e^{kx}$. Shew that an eye placed in this medium at a point x, y will see the origin by means of a small pencil, one of whose focal lines is on the axis of x, and the other at distance v from the eye, where v is given by the equation

$$k^2 v^2 \cos^2 ky = 2 (\cosh kx - \cos ky)\, e^{kx},$$

while the axis of the pencil makes an angle ϕ with the axis of x given by the equation

$$\cot \phi \sin ky = e^{kx} - \cos ky.$$

7. Find the path of a ray in a medium in which μ is proportional to $(a^2+r^2)^{\frac{1}{2}}/r$; and shew that in certain cases it is a reciprocal spiral.

8. Prove that if μ varies as $r^{-(n+1)}$, the paths of rays in this medium and their orthotomic curves are given by the equation

$$U+iV=r^{-n}\{\cos(n\theta-\omega)-i\sin(n\theta-\omega)\}.$$

9. If a refracting medium be arranged in spherical concentric strata, and the paths be coaxial lemniscates with the centre as pole, shew that

$$\mu=r^3 f(r^2 \operatorname{cosec} 2\theta).$$

10. Prove that if μ varies as $(a^2+r^2)^{-1}$ where a is a constant, the path of every ray is a circle, and that all rays whatever through a given point meet again in a point on the radius through the origin.

Prove that in such a medium the linear magnification is the ratio of the distances of the image and object from the origin.

11. A medium is stratified cylindrically so that $\mu=a/r$, where r is the distance of any point from the axis of z, and a ray is incident at distance a from that axis, the angle of incidence being ϕ, and the plane of incidence making an angle ψ with the meridian plane through the axis. Shew that the path is the intersection of a sphere with the cylinder

$$\frac{a}{r}=\frac{\sin\phi\cos\psi}{(1-\sin^2\phi\sin^2\psi)^{\frac{1}{2}}}\cosh\left\{\frac{(1-\sin^2\phi\sin^2\psi)^{\frac{1}{2}}}{\sin\phi\sin\psi}\theta\right\}.$$

12. A prism is such that the refractive index of a plane section through the edge inclined at an angle θ to one of the faces is $\mu_0 e^{-\theta\tan a}$. A ray incident perpendicularly on this face at distance a from the edge passes in a principal plane; shew that when the angle of incidence on a plane section of the prism is ϕ, the coordinates of the point of incidence are given by

$$r/a=e^{\phi\sin a\cos a}\{\cos a\sec(\phi-a)\}^{\cos^2 a},$$
$$\theta=\phi\cos^2 a-\sin a\cos a\log\{\cos a\sec(\phi-a)\}.$$

13. A medium is bounded by the planes of x and y, the refractive index at any point being given by $\log\mu=xy/a^2$; two rays are incident on it respectively parallel to the axes and at equal distances c from the origin; shew that if they intersect, it is at an angle $\frac{1}{2}\pi-c^2/a^2$.

14. A ray passes in the plane of xy, through a heterogeneous medium; shew that the projection of the radius of curvature of the path at any point on the normal to the refracting surface at that point is

$$\mu\left\{\left(\frac{\partial\mu}{\partial x}\right)^2+\left(\frac{\partial\mu}{\partial y}\right)^2\right\}^{-\frac{1}{2}}.$$

15. The two-dimensional path of a ray is referred to conjugate functions a, β of the coordinates x, y. Shew that the equation of the path is

$$\mu h^3\left(\frac{da}{ds}\frac{d^2\beta}{ds^2}-\frac{d\beta}{ds}\frac{d^2a}{ds^2}\right)=\frac{da}{ds}\frac{\partial}{\partial\beta}(\mu h)-\frac{d\beta}{ds}\frac{\partial}{\partial a}(\mu h);$$

and that the curvature is

$$\left[\left\{\frac{1}{h}\frac{d}{ds}\left(h^2\frac{da}{ds}\right)-\frac{1}{h^2}\frac{\partial h}{\partial a}\right\}^2+\left\{\frac{1}{h}\frac{d}{ds}\left(h^2\frac{d\beta}{ds}\right)-\frac{1}{h^2}\frac{\partial h}{\partial\beta}\right\}^2\right]^{\frac{1}{2}},$$

where $$(ds)^2\equiv h^2\{(da)^2+(d\beta)^2\}.$$

16. Prove that if an image of a small object at distance r from the centre of a medium, in which $\mu=f(r)$, be formed at distance r' by convergence of rays in a plane containing the centre, the magnification measured in the plane of convergence perpendicular to the axis of the pencil is $\dfrac{\mu\tan\phi}{\mu'\tan\phi'}$ and along the axis of the pencil is $\dfrac{\mu\tan^2\phi}{\mu'\tan^2\phi'}$, where the axial ray leaves the object in a direction inclined to r at an angle ϕ, and reaches the image in a direction inclined to r' at angle ϕ'.

17. If μ^2 varies as (c^2-r^2) the path of any ray is an ellipse, with its centre at the origin; and the envelope of all rays passing through a given point is a prolate spheroid, having the given point as a focus.

Shew that the distance from the origin of the point where a given ray touches the envelope is $[\{c^4-(2c^2-r^2)r^2\sin^2\phi\}/(c^2-r^2\sin^2\phi)]^{\frac{1}{2}}$, where r, ϕ refer to the origin of the pencil.

18. Deduce from Fermat's principle the change in the law of refractive index which would accompany the inversion by reciprocal radii vectores of a system of ray paths.

Shew that in a heterogeneous medium in which μ varies as r^{-2}, the images of all points formed by spherical reflectors through the origin are perfect images.

19. The torsion of the path of a ray in a heterogeneous medium is

$$\begin{vmatrix} V_x, & V_y, & V_z \\ \mu_x, & \mu_y, & \mu_z \\ \dfrac{d\mu_x}{ds}, & \dfrac{d\mu_y}{ds}, & \dfrac{d\mu_z}{ds} \end{vmatrix} \Bigg/ \mu \left\{\mu_x^2+\mu_y^2+\mu_z^2-\left(\frac{d\mu}{ds}\right)^2\right\},$$

where $V=\int\mu ds$, and the suffixes denote partial differentiation with regard to the variables.

CHAPTER XIV.

DISTORTION.

233. THE formulae obtained in Chapter VIII. for the aberration in a coaxial symmetrical instrument were only applicable to pencils diverging from points on the axis. Of course in the case of a single refraction the radius through the origin of light would be regarded as the axis, but that method cannot be extended to further refractions.

In the present chapter we develop formulae by which we can determine the point in which any given ray from a point of a small object perpendicular to the axis crosses the plane of the geometrical foci, when squares of the distances from the axis and other quantities previously neglected are retained. The emergent pencil meets this focal plane in a small area surrounding the geometrical focus of the origin of light; and the greatest diameter of this area may be taken to measure the *indistinctness.*

Next, concentrating our attention on one chief ray of the pencil, which may be chosen as passing through the centre of a diaphragm in photographic apparatus, or as ultimately entering the eye in an instrument adapted for vision, we take the point in which this ray crosses the focal plane as the *apparent* image. The ratio between the distances of this point and of the geometrical focus from the axis will differ from unity by a function of the distance from the axis, which will be taken to measure the *distortion.*

Again, this ray will be met by an adjacent ray lying in the same axial plane in the primary focus, and by an adjacent ray lying in the perpendicular plane in the secondary focus; and the

curvatures of the images of the original small object may be calculated, as they are formed by primary or secondary foci. The magnitudes of these curvatures will measure *the flatness of the field.*

The small pencil of rays adjacent to the chief ray will pass through a circle of least *confusion*; and the nearer to each other the primary and secondary foci lie, the smaller will be this circle of confusion.

234. Let a spherical surface of radius ρ separate two media of indices μ and μ'; let the origin of coordinates be on the surface, and let the axis of z be a radius, as in Art. 162. Let (x, y, z) be the coordinates of an origin of light near the axis of z, and (l, m, n) the direction-cosines of a ray nearly parallel to the axis. Then in the following x, y, l, m are small quantities of the first order, whose squares and products are to be retained, while higher powers are neglected, and z is finite. Let (ξ, η, ζ) be the coordinates of the point of incidence of the ray, and R the distance from the origin of light, so that

$$\xi = x + lR, \quad \eta = y + mR, \quad \zeta = z + nR.$$

Also to the second order

$$\zeta = (\xi^2 + \eta^2)/2\rho, \text{ and } n = -\{1 - \tfrac{1}{2}(l^2 + m^2)\}.$$

Hence $\quad R = z + \frac{1}{2}\{z(l^2 + m^2) - (\xi^2 + \eta^2)/\rho\} = z(1 + u)\ldots\text{(i)},$

where $\quad u = \frac{1}{2}\{(l^2 + m^2) - (\xi^2 + \eta^2)/z\rho\}.$

Also $\quad \xi = x + lz(1 + u), \quad \eta = y + mz(1 + u).$

The angle of incidence of the ray being ϕ, we have

$$\begin{aligned}\cos\phi &= \{l\xi + m\eta + n(\zeta - \rho)\}/\rho \\ &= [l\xi + m\eta - (\zeta - \rho)\{1 - \tfrac{1}{2}(l^2 + m^2)\}]/\rho \\ &= 1 - \frac{1}{2}\left\{\frac{\xi^2 + \eta^2}{\rho^2} - 2\frac{l\xi + m\eta}{\rho} + l^2 + m^2\right\} = 1 - \delta\ldots\text{(ii)}.\end{aligned}$$

Again, if the ray after refraction pass through a point (x', y', z') and its direction-cosines be (l', m', n'), we have in the same way

$$\xi = x' + l'z'(1 + u'), \quad \eta = y' + m'z'(1 + u'),$$
$$\cos\phi' = 1 - \delta',$$

where u', δ' denote quantities similar to u and δ.

By Art. 18 the relation between the direction-cosines is

$$\mu' l' - \mu l = (\mu' \cos \phi' - \mu \cos \phi)\, \xi/\rho$$
$$= \{\mu'(1 - \delta') - \mu(1 - \delta)\}\, \xi/\rho \quad\text{.........(iii).}$$

The first approximation to the solution of these equations being obtained by neglecting the squares, we have

$$\left.\begin{aligned} \xi &= x + lz = x' + l'z' \\ \mu' l' - \mu l &= (\mu' - \mu)\, \xi/\rho \end{aligned}\right\} \quad\text{..................(iv).}$$

If z' be chosen so that $(\mu' - \mu)/\rho = \mu'/z' - \mu/z$, *i.e.* if we find the point at which the ray cuts the plane drawn through the geometrical focus perpendicular to the axis, the second of equations (iv) may be written

$$\mu' l' - \mu l = \frac{\mu'}{z'}(x' + l'z') - \frac{\mu}{z}(x + lz).$$

Hence $\dfrac{\mu x}{z} = \dfrac{\mu' x'}{z'}$, and $\dfrac{\mu y}{z} = \dfrac{\mu' y'}{z'}$, the ordinary formulae for the linear magnification at the geometrical focus.

For the second approximation let $x + \mathbf{d}x$, $x' + \mathbf{d}x'$, &c. be the solutions of (iii) when squares are retained. Then

$$\mathbf{d}\xi = \mathbf{d}x' + z'\mathbf{d}l' + l'z'u' = \mathbf{d}x + z\mathbf{d}l + lzu \quad\text{..........(v),}$$

and
$$\mu'\mathbf{d}l' - \mu\mathbf{d}l = \frac{(\mu' - \mu)}{\rho}\mathbf{d}\xi - (\mu'\delta' - \mu\delta)\frac{\xi}{\rho}$$
$$= \frac{\mu'}{z'}(\mathbf{d}x' + z'\mathbf{d}l' + l'z'u') - \frac{\mu}{z}(\mathbf{d}x + z\mathbf{d}l + lzu)$$
$$- (\mu'\delta' - \mu\delta)\, \xi/\rho,$$

or
$$\frac{\mu'}{z'}\mathbf{d}x' - \frac{\mu}{z}\mathbf{d}x = (\mu'\delta' - \mu\delta)\frac{\xi}{\rho} - (\mu' l' u' - \mu l u).$$

Substituting the values of u and δ, u' and δ', this gives

$$\frac{\mu'}{z'}\mathbf{d}x' - \frac{\mu}{z}\mathbf{d}x = (\mu' - \mu)\frac{\xi}{\rho}\frac{\xi^2 + \eta^2}{2\rho^2} - \left\{(\mu' l' - \mu l)\frac{\xi}{\rho} + (\mu' m' - \mu m)\frac{\eta}{\rho}\right\}\frac{\xi}{\rho}$$
$$+ \frac{\mu'}{2}\left(\frac{\xi}{\rho} - l'\right)(l'^2 + m'^2) - \frac{\mu}{2}\left(\frac{\xi}{\rho} - l\right)(l^2 + m^2)$$
$$+ \left(\frac{\mu' l'}{z'} - \frac{\mu l}{z}\right)\frac{(\xi^2 + \eta^2)}{2\rho} \quad\text{........................ (vi).}$$

On the right-hand side the first approximations may be substituted, so that the second term is twice the first term. To convert this equation into a difference-formula, and to remove the distances z and z' from an origin, which in the case of coaxial

surfaces would need to be changed at each refraction, consider a ray, diverging from the focus on the axis at a small arbitrary angle α, and meeting the refracting surface at distance r from the axis. Then $r = z\alpha = z'\alpha'$, and $\mu'\alpha' - \mu\alpha = (\mu' - \mu) r/\rho$.

Multiply equation (vi) throughout by r, and make the substitutions $\mu'\left(\frac{\xi}{\rho} - l'\right) = \mu\left(\frac{\xi}{\rho} - l\right) = \frac{\mu\mu'}{\mu' - \mu}(l' - l)$; we then obtain

$$\mu'\alpha'\mathbf{d}x' - \mu\alpha\mathbf{d}x = \frac{\mu\mu'}{\mu' - \mu}(l' - l)(l'^2 + m'^2 - l^2 - m^2)\frac{r}{2}$$

$$- \frac{\xi^2 + \eta^2}{2\rho}\left\{(\mu' - \mu)\frac{r\xi}{\rho^2} - (\mu' l'\alpha' - \mu l\alpha)\right\},$$

$$= \frac{1}{2}\frac{\mu\mu'}{\mu' - \mu}(l' - l)\left\{(l'^2 + m'^2 - l^2 - m^2) r - \frac{\xi^2 + \eta^2}{\rho}(\alpha' - \alpha)\right\} \quad \text{...(vii)},$$

on substituting for r and ξ in the second line.

If we substitute for the first powers of ξ and η from (iv), and write the term $(\mu' - \mu)(l'^2 - l^2)$ in the form

$$(l' - l)\left\{\mu' l' - \mu l - \mu\mu'\left(\frac{l'}{\mu'} - \frac{l}{\mu}\right)\right\},$$

we finally obtain for $\mu'\alpha'\mathbf{d}x' - \mu\alpha\mathbf{d}x$ the expression

$$-\frac{1}{2}\frac{\mu\mu'}{(\mu' - \mu)^2}(l' - l)\left[\mu\mu' r\left\{(l' - l)\left(\frac{l'}{\mu'} - \frac{l}{\mu}\right) + (m' - m)\left(\frac{m'}{\mu'} - \frac{m}{\mu}\right)\right\}\right.$$

$$\left. + (\mu' l' - \mu l)\begin{vmatrix} \xi, & r \\ l' - l, & \alpha' - \alpha \end{vmatrix} + (\mu' m' - \mu m)\begin{vmatrix} \eta, & r \\ m' - m, & \alpha' - \alpha \end{vmatrix}\right] \quad \text{......(A)}.$$

This formula includes as a particular case the aberration formula (A) of Art. 141. For if the origin of light be on the axis we may take η, m and m' zero, ξ equal to r, l to α, and l' to α'. Also $x' + \mathbf{d}x' = -\alpha'\mathbf{d}v$ with the notation of that article, while by Helmholtz's formula $\mu'\alpha' x' = \mu\alpha x$ as a first approximation, so that $\mu'\alpha'\mathbf{d}x' - \mu\alpha\mathbf{d}x = -(\mu'\alpha'^2\mathbf{d}v - \mu\alpha^2\mathbf{d}u)$.

235. Successive refractions.

When the ray is refracted successively at coaxial spherical surfaces, separating media of indices $\mu_0, \mu_1 \ldots \mu_n, \ldots$ and at intervals $\mu_1\tau_1, \mu_2\tau_2, \ldots$ apart, the first approximations to its direction-cosines in any of the media are given by the difference-equations

$$\mu_n l_n - \mu_{n-1} l_{n-1} = \kappa_n \xi_n,$$

$$\xi_n - \xi_{n-1} = \mu_{n-1} l_{n-1} \tau_{n-1}.$$

These equations are of the same form as those for the angle of divergence and the apparent distance (Arts. 75, 86), and hence

$$\left.\begin{aligned} \mu_n l_n &= K_n \xi_1 + \frac{\partial K_n}{\partial \kappa_1} \mu_0 l_0 \\ \xi_n &= \frac{\partial K_n}{\partial \kappa_n} \xi_1 + \frac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n} \mu_0 l_0 \end{aligned}\right\} \qquad \text{(viii)},$$

where K_n is the power of the first n refracting surfaces.

We have also for the quantities r and α used in (A) the equations

$$\left.\begin{aligned} \frac{\mu_n \alpha_n}{\mu_0 \alpha_0} &= K_n \frac{z}{\mu_0} + \frac{\partial K_n}{\partial \kappa_1} \\ \frac{r_n}{\mu_0 \alpha_0} &= \frac{\partial K_n}{\partial \kappa_n} \frac{z}{\mu_0} + \frac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n} \end{aligned}\right\} \qquad \text{(ix)}.$$

Hence by addition of n equations of the type (A) (suffixed letters in succession replacing the corresponding symbols of that formula), we can finally determine $\mathbf{d}x_n$, $\mathbf{d}y_n$ as cubic functions of ξ_1, η_1, l_0 and m_0, in which the coefficients involve z, and thus find to this second approximation the point in which any given ray cuts the plane through the final geometrical focus.

236. The factor that occurs on the right-hand side of (A) may be considerably simplified.

First, if we substitute in (viii) for ξ_1, these equations may be rewritten as

$$\mu_n l_n = K_n (x + l_0 z) + \frac{\partial K_n}{\partial \kappa_1} \mu_0 l_0 = K_n x + l_0 \mu_n \alpha_n / \alpha_0 \qquad \text{(viii)}',$$

$$\xi_n = \frac{\partial K_n}{\partial \kappa_n} (x + l_0 z) + \frac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n} \mu_0 l_0 = \frac{\partial K_n}{\partial \kappa_n} x + l_0 r_n / \alpha_0 \qquad \text{(viii)}''.$$

Secondly, between the powers of the systems, consisting of $(n-1)$ and n surfaces, and their differential coefficients we have the identities (cf. Art. 66)

$$\left.\begin{aligned} \frac{\partial K_n}{\partial \kappa_1} \frac{\partial K_n}{\partial \kappa_n} - K_n \frac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n} &\equiv 1 \\ K_n - K_{n-1} &\equiv \kappa_n \frac{\partial K_n}{\partial \kappa_n} \\ \frac{\partial K_n}{\partial \kappa_1} - \frac{\partial K_{n-1}}{\partial \kappa_1} &\equiv \kappa_n \frac{\partial^2 K_n}{\partial \kappa_1 \partial \kappa_n} \end{aligned}\right\} \qquad \text{(x)},$$

whence

$$K_n \frac{\partial K_{n-1}}{\partial \kappa_1} - K_{n-1} \frac{\partial K_n}{\partial \kappa_1} \equiv \kappa_n.$$

In this and the following articles let ΔK_n denote $K_n - K_{n-1}$, with a similar notation for the other quantities involved, as μ, l, r, and α.

The term $\Delta(\mu_n l_n)\begin{vmatrix} \xi_n, & r_n \\ \Delta l_n, & \Delta\alpha_n \end{vmatrix}$, which occurs in (A), is equal to

$$\kappa_n\xi_n \begin{vmatrix} x\dfrac{\partial K_n}{\partial\kappa_n} + l_0\dfrac{r_n}{\alpha_0}, & r_n \\ x\Delta\left(\dfrac{K_n}{\mu_n}\right) + l_0\dfrac{\Delta\alpha_n}{\alpha_0}, & \Delta\alpha_n \end{vmatrix}$$

$$= \kappa_n\xi_n x\alpha_0 \begin{vmatrix} \dfrac{\partial K_n}{\partial\kappa_n}, & z\dfrac{\partial K_n}{\partial\kappa_n} + \mu_0\dfrac{\partial^2 K_n}{\partial\kappa_1\partial\kappa_n} \\ \Delta\left(\dfrac{K_n}{\mu_n}\right), & z\Delta\left(\dfrac{K_n}{\mu_n}\right) + \mu_0\Delta\left(\dfrac{1}{\mu_n}\dfrac{\partial K_n}{\partial\kappa_1}\right) \end{vmatrix} \text{ from (ix),}$$

$$= \mu_0\alpha_0\xi_n x \begin{vmatrix} \Delta K_n, & \Delta\left(\dfrac{\partial K_n}{\partial\kappa_1}\right) \\ \Delta\left(\dfrac{K_n}{\mu_n}\right), & \Delta\left(\dfrac{1}{\mu_n}\dfrac{\partial K_n}{\partial\kappa_1}\right) \end{vmatrix} \text{ from (x),}$$

$$= \mu_0\alpha_0\xi_n x \begin{vmatrix} K_n, & K_{n-1} \\ \dfrac{\partial K_n}{\partial\kappa_1}, & \dfrac{\partial K_{n-1}}{\partial\kappa_1} \end{vmatrix} \begin{vmatrix} 1, & -1 \\ \dfrac{1}{\mu_n}, & -\dfrac{1}{\mu_{n-1}} \end{vmatrix}$$

$$= -\mu_0\alpha_0\xi_n x\kappa_n\frac{\Delta\mu_n}{\mu_n\mu_{n-1}}.$$

Hence (A) may be written in the form

$$\Delta(\mu_n\alpha_n \mathbf{d}x_n) = -\frac{1}{2}\frac{\Delta l_n}{\Delta\mu_n}\left[\frac{\mu_n^2\mu_{n-1}^2}{\Delta\mu_n}\left\{\Delta l_n\Delta\left(\frac{l_n}{\mu_n}\right) + \Delta m_n\Delta\left(\frac{m_n}{\mu_n}\right)\right\} r_n \right.$$
$$\left. - \mu_0\alpha_0\kappa_n(\xi_n x + \eta_n y)\right] \dots\dots \text{(B).}$$

When it is necessary to express $\mathbf{d}x_n$ in terms of the initial point of incidence (ξ_1, η_1), this form will be found to be the simplest for summation.

237. *To find the distribution of light on the second focal plane of a thin lens.*

If we consider an origin of light at infinity in the plane of zx at angular distance θ from the axis, then to the first approximation the rays are brought to a focus at a point on the second focal plane at distance $x_2 (= -f\theta)$ from the axis. Proceeding to calculate the quantities involved in the difference-equation (B) applied to the two refractions, we have $l_0 = \theta$, $m_0 = 0$ for all the

incident parallel rays; and if (R, χ) be the coordinates of the point of incidence of a ray on the lens,

$$\xi = R\cos\chi, \quad \eta = R\sin\chi \text{ throughout.}$$

If κ_1 be the power of the first surface and K the power of the lens, the direction-cosines of this ray in the lens are given by

$$\mu l_1 = \kappa_1 R\cos\chi + \theta, \quad \mu m_1 = \kappa_1 R\sin\chi,$$

and on emergence by

$$l_2 = KR\cos\chi + \theta, \quad m_2 = KR\sin\chi.$$

For the auxiliary quantities a_0, a_1, a_2 and r, we have

$$\mu a_1/a_0 = 1 + \kappa_1 z, \quad a_2/a_0 = 1 + Kz,$$

where a_0 is zero and z infinite in such a manner that $za_0 = r$. The coordinate x of the extra-axial origin of light is $-z\theta$, and hence $a_0 x = -r\theta$.

If these values be substituted in (B) applied to the two refractions, and the algebraical reductions performed, we obtain

$$\mathbf{d}x_2 = -R\cos\chi\,\mathfrak{A} + x_2\mathfrak{B}, \quad \mathbf{d}y_2 = -R\sin\chi\,\mathfrak{A},$$

where

$$\mathfrak{A} = R^2\frac{\mu^3K^2 - (2\mu^2+\mu)K\kappa_1 + (\mu+2)\kappa_1^2}{2\mu(\mu-1)^2} - Rx_2\cos\chi\frac{\mu^2K^2 - (\mu+1)K\kappa_1}{\mu(\mu-1)} + x_2^2\frac{\mu+1}{2\mu}K^2,$$

$$\mathfrak{B} = R^2\frac{\mu^2K^2 - (\mu+1)K\kappa_1}{2\mu(\mu-1)} - Rx_2\cos\chi\,K^2.$$

The first term in $\mathfrak{A}$ is of course the aberration-formula; and if the incident rays fill the whole aperture of the lens, those which are incident parallel to the axis cross the second focal plane on emergence in a small circle, while those which are incident in the direction θ cross the focal plane within a small *area of indistinctness*, having its centre at x_2 and given by the equations above.

The radius of the circle surrounding the principal focus is four times the radius of the least circle of aberration; similarly any section of the oblique pencil near the focal plane will be of the same order of smallness as this area of indistinctness, which is a quartic curve, reducing to an ellipse if $\mu^2K = (\mu+1)\kappa_1$, and this is the nearest approach it makes to a circle. If μ be such that $\mu^2 = (\mu+1)$, or $\mu = \frac{1}{2}(1+\sqrt{5}) = 1{\cdot}62$, then in this case the lens is convexo-plane, and the semi-axes of the ellipse are approximately $R(\cdot 8R^2 + 1{\cdot}8x_2^2)/f^2$ in the axial plane through the origin of light, and $R(\cdot 8R^2 + \cdot 8x_2^2)/f^2$ perpendicular to this plane. It is a striking fact that this value of μ is not far removed from that value for which a convexo-plane lens gives minimum aberration at the principal focus (Art. 147); and hence a telescopic lens, if corrected for aberration at its second focus, will also give a distinct image all over its focal plane, for in such a lens x_2 is certainly small compared with R.

The case of a single photographic landscape lens is however quite different; here a stop is always placed at a certain distance in front of the lens, so that the image on the plate is formed at its centre by rays through the stop and

the central points of the lens, and towards its edges by rays through the stop and the marginal parts of the lens. The area of indistinctness on the plate is part only of the quartic curve, and would be found by substituting above $R_0 \cos\omega + \theta d$ for $R\cos\chi$, and $R_0 \sin\omega$ for $R\sin\chi$, where R_0 is the radius of aperture of the stop, d its distance from the lens. To obtain a photograph distinct all over, the coefficients of x_2 and x_2^2 in the values obtained for $\mathbf{d}x_2$ and $\mathbf{d}y_2$ should be as small as possible, for here the extreme values of x_2 are certainly many multiples of R_0.

238. In obtaining the curvature of the image it is necessary to further transform the result obtained in (B).

Substitute from (viii)′ in the product $\Delta(l_n)\Delta(l_n/\mu_n)$; we have

$$\Delta l_n \Delta\left(\frac{l_n}{\mu_n}\right) = \left\{x\Delta\left(\frac{K_n}{\mu_n}\right) + \frac{l_0}{\alpha_0}\Delta\alpha_n\right\}\left\{x\Delta\left(\frac{K_n}{\mu_n^2}\right) + \frac{l_0}{\alpha_0}\Delta\left(\frac{\alpha_n}{\mu_n}\right)\right\};$$

and in this the coefficient of $\dfrac{xl_0}{\alpha_0}$ is $\Delta\alpha_n \Delta\left(\dfrac{K_n}{\mu_n^2}\right) + \Delta\left(\dfrac{\alpha_n}{\mu_n}\right)\Delta\left(\dfrac{K_n}{\mu_n}\right)$,

or $$2\Delta\left(\frac{\alpha_n}{\mu_n}\right)\Delta\left(\frac{K_n}{\mu_n}\right) + \begin{vmatrix} \Delta\alpha_n, & \Delta(\alpha_n/\mu_n) \\ \Delta(K_n/\mu_n), & \Delta(K_n/\mu_n^2) \end{vmatrix}.$$

But this determinant

$$= \begin{vmatrix} \alpha_n, & \alpha_{n-1} \\ K_n/\mu_n, & K_{n-1}/\mu_{n-1} \end{vmatrix} \begin{vmatrix} 1, & -1 \\ 1/\mu_n, & -1/\mu_{n-1} \end{vmatrix}$$

$$= \frac{-\alpha_0\Delta\mu_n}{\mu_n^2\mu_{n-1}^2} \begin{vmatrix} zK_n + \mu_0\dfrac{\partial K_n}{\partial\kappa_1}, & zK_{n-1} + \mu_0\dfrac{\partial K_{n-1}}{\partial\kappa_1} \\ K_n, & K_{n-1} \end{vmatrix}$$

$$= \mu_0\alpha_0\kappa_n\Delta\mu_n/\mu_n^2\mu_{n-1}^2, \quad \text{from (x).}$$

Hence (B) may be written in the form

$$\Delta(\mu_n\alpha_n\mathbf{d}x_n) = -\frac{1}{2}\frac{\Delta l_n}{\Delta\mu_n}\left[\frac{\mu_n^2\mu_{n-1}^2}{\Delta\mu_n}\left\{(x^2+y^2)\Delta\left(\frac{K_n}{\mu_n}\right)\Delta\left(\frac{K_n}{\mu_n^2}\right)\right.\right.$$
$$\left. + 2\frac{xl_0+ym_0}{\alpha_0}\Delta\left(\frac{\alpha_n}{\mu_n}\right)\Delta\left(\frac{K_n}{\mu_n}\right) + \frac{l_0^2+m_0^2}{\alpha_0^2}\Delta\alpha_n\Delta\left(\frac{\alpha_n}{\mu_n}\right)\right\}r_n$$
$$\left. - \mu_0\alpha_0\kappa_n\left\{\left(\xi_n - \frac{l_0}{\alpha_0}r_n\right)x + \left(\eta_n - \frac{l_0}{\alpha_0}r_n\right)y\right\}\right].$$

In the first bracket on the right complete the square in l_0/α_0; we obtain

$$\frac{\Delta\left(\dfrac{\alpha_n}{\mu_n}\right)}{\Delta\alpha_n}\left\{x\Delta\left(\frac{K_n}{\mu_n}\right) + \frac{l_0}{\alpha_0}\Delta\alpha_n\right\}^2 + x^2\frac{\Delta\left(\dfrac{K_n}{\mu_n}\right)}{\Delta\alpha_n}\begin{vmatrix} \Delta\alpha_n, & \Delta\left(\dfrac{\alpha_n}{\mu_n}\right) \\ \Delta\left(\dfrac{K_n}{\mu_n}\right), & \Delta\left(\dfrac{K_n}{\mu_n^2}\right) \end{vmatrix},$$

i.e.
$$\frac{\Delta(\alpha_n/\mu_n)}{\Delta\alpha_n}(\Delta l_n)^2 + x^2\frac{\Delta(K_n/\mu_n)}{\Delta\alpha_n}\frac{\mu_0\alpha_0\kappa_n\Delta\mu_n}{\mu_n^2\mu^2_{n-1}}.$$

Moreover $\xi_n - \dfrac{l_0}{\alpha_0}r_n = x\dfrac{\partial K_n}{\partial\kappa_n}$ from (viii)″; hence the coefficient of x^2 in the entire square bracket is

$$\frac{\mu_0\alpha_0\kappa_n}{\Delta\alpha_n}\left\{r_n\Delta\left(\frac{K_n}{\mu_n}\right) - \Delta\alpha_n\frac{\partial K_n}{\partial\kappa_n}\right\}$$

$$= -\frac{\mu_0\alpha_0^2\kappa_n}{\Delta\alpha_n}\begin{vmatrix} \dfrac{\partial K_n}{\partial\kappa_n}, & z\dfrac{\partial K_n}{\partial\kappa_n} + \mu_0\dfrac{\partial^2 K_n}{\partial\kappa_1\partial\kappa_n} \\ \Delta\left(\dfrac{K_n}{\mu_n}\right), & z\Delta\left(\dfrac{K_n}{\mu_n}\right) + \mu_0\Delta\left(\dfrac{1}{\mu_n}\dfrac{\partial K_n}{\partial\kappa_1}\right)\end{vmatrix}$$

$$= \frac{\mu_0^2\alpha_0^2\kappa_n\Delta\mu_n}{\mu_n\mu_{n-1}\Delta\alpha_n}, \text{ as in Art. 236.}$$

We therefore obtain finally

$$\Delta(\mu_n\alpha_n\mathbf{d}x_n) = -\frac{1}{2}\frac{\Delta l_n}{\Delta\alpha_n}\left[\left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2 r_n\Delta\left(\frac{\alpha_n}{\mu_n}\right)\{(\Delta l_n)^2 + (\Delta m_n)^2\}\right.$$
$$\left. + \frac{\mu_0^2\alpha_0^2}{\mu_n\mu_{n-1}}\kappa_n(x^2+y^2)\right]\ldots\ldots\ldots\ldots(\text{C}).$$

In all these formulae, (A), (B) and (C), the quantities r and α occur in the first degree on both sides, and are only fictitious quantities introduced in place of the distance of the origin of light from the first surface. If we substitute in (C) for l_n and m_n in terms of the coordinates x, y of the origin of light and the initial direction-cosines l, m of the ray, we may write the formula in the form

$$\frac{\Delta(\mu_n\alpha_n\mathbf{d}x_n)}{\mu_0\alpha_0} = -\frac{1}{2}\left[\left(u_1 x + \frac{\mu_0\kappa_n}{\mu_n\mu_{n-1}}l\right)(x^2+y^2)\right.$$
$$+ u_2\{2x(lx+my) + l(x^2+y^2)\}$$
$$\left. + u_3\{x(l^2+m^2) + 2l(lx+my)\} + u_4 l(l^2+m^2)\right]\ldots(\text{C}'),$$

where

$$u_1 = \left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2\left(\Delta\frac{K_n}{\mu_n}\right)^2\Delta\left(\frac{K_n}{\mu_n^2}\right)\frac{r_n}{\mu_0\alpha_0} - \Delta\left(\frac{K_n}{\mu_n}\right)\frac{\Delta K_n}{\Delta\mu_n},$$

$$u_2 = \left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2\left(\Delta\frac{K_n}{\mu_n}\right)^2\Delta\left(\frac{\alpha_n}{\mu_n}\right)\frac{r_n}{\mu_0\alpha_0^2},$$

$$u_3 = \left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2\Delta\left(\frac{K_n}{\mu_n}\right)\Delta\alpha_n\Delta\left(\frac{\alpha_n}{\mu_n}\right)\frac{r_n}{\mu_0\alpha_0^3},$$

$$u_4 = \left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2(\Delta\alpha_n)^2\Delta\left(\frac{\alpha_n}{\mu_n}\right)\frac{r_n}{\mu_0\alpha_0^4}.$$

The suffixes to u_1, .. u_4 indicate the degree to which the coordinate z of the origin of light enters; the value of u_1 was found by substitution in (B) instead of (C). When the origin of light is on the axis, x and y are zero, and the only term in (C′) is u_4, which is the aberration-formula of Art. 141.

239. Distortion.

The above articles shew plainly that when the origin of light lies off the axis of an instrument, and the rays make a small but finite angle with the axis, there is no definite image. If however the eye lie on the axis of the instrument, we may perhaps regard the point in which the ray that enters the focal centre of the eye cuts the plane of the geometrical foci as the apparent image. As the system is symmetrical about its axis, this ray meets the axis in all parts of its path, and we may take y and m zero in this case.

Hence the image of a small object of height x appears of height $x_n + \mathbf{d}x_n$; and the linear magnification is changed from its value for the geometrical focus in the ratio $1 + \mathbf{d}x_n/x_n : 1$. The distortion may therefore be measured by $\mathbf{d}x_n/x_n$, or, since $\mu_n a_n x_n = \mu_0 a_0 x$, by $\mu_n a_n \mathbf{d}x_n/\mu_0 a_0 x$.

Putting m and y zero, and performing the summation for the n refractions, we obtain from (C′) the measure of the distortion in the form

$$-\frac{1}{2}\left[lx\sum_1^n \frac{\mu_0\kappa_r}{\mu_r\mu_{r-1}} + \frac{U_1x^3 + 3U_2x^2l + 3U_3xl^2 + U_4l^3}{x}\right]\ldots(\text{D}),$$

where U_1, U_2, .. are written for Σu_1, Σu_2, ...

If this quantity be positive, the image of a small network of squares (fig. i) will be distorted into the 'pin-cushion' form (fig. ii), if it be negative, into the 'barrel' form (fig. iii).

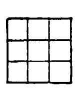


Fig. i.

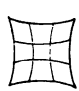


Fig. ii.

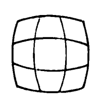


Fig. iii.

240. Curvature of Image.

On every ray of the emergent pencil there are two focal points in which it is intersected by adjacent rays. If as in the previous article we confine our attention to a certain ray which lies in the axial plane throughout, this ray will be met by an adjacent ray

also in the axial plane in the primary focus; and by a ray diverging in the perpendicular plane in the secondary focus.

First, for rays in the axial plane, we have m and y zero; also

$$\mu_n l_n = K_n x + l\mu_n \alpha_n/\alpha_0.$$

Hence if x and z remain unaltered while l varies, we have

$$\frac{\partial l_n}{\partial l} = \frac{\alpha_n}{\alpha_0}, \text{ and } \frac{\partial(\Delta l_n)}{\partial l} = \frac{\Delta\alpha_n}{\alpha_0}.$$

The distance of the point of intersection of consecutive rays from the focal plane will be $\frac{\partial}{\partial l_n}(\mathbf{d}x_n)$; and the ordinate of this point is approximately x_n, or $\mu_0\alpha_0 x/\mu_n\alpha_n$.

If then R_n be the radius of curvature of the nth image formed by primary foci, reckoned positive when convex to the eye, we have by Newton's formula

$$\frac{1}{R_n} = \frac{2}{x_n^2}\frac{\partial}{\partial l_n}(\mathbf{d}x_n) = 2\frac{\mu_n^2\alpha_n}{\mu_0^2\alpha_0 x^2}\frac{\partial}{\partial l}(\mathbf{d}x_n);$$

and therefore

$$\Delta\left(\frac{1}{\mu_n R_n}\right) = \frac{2}{\mu_0^2\alpha_0 x^2}\frac{\partial}{\partial l}\Delta(\mu_n\alpha_n\mathbf{d}x_n).$$

Differentiating (C) with respect to l, we obtain

$$\Delta\left(\frac{1}{\mu_n R_n}\right) = -\left\{\frac{\kappa_n}{\mu_n\mu_{n-1}} + 3\left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2 \frac{r_n\Delta\left(\frac{\alpha_n}{\mu_n}\right)}{\mu_0^2\alpha_0^2}\frac{(\Delta l_n)^2}{x^2}\right\} \ldots(\text{E}),$$

$$= -\{s+3t\},$$

or if we take the form given in (C′) and perform the summations, we may obtain the curvature of the final image in the form

$$\frac{1}{\mu_n R_n} = -\left\{\sum_1^n \frac{\kappa_r}{\mu_r\mu_{r-1}} + 3\,\frac{U_2x^2 + 2U_3lx + U_4l^2}{x^2}\right\} \ldots\ldots(\text{E}'),$$

$$= -\{S+3T\}.$$

Again, for the curvature of the image formed by secondary foci, we consider the adjacent ray which makes a small angle m with the axial plane; for this ray $m_n = m\alpha_n$, and the value of $\mathbf{d}x_n$ differs from its value when m is zero by squares only of m. But since $\Delta m_n = m\Delta\alpha_n$, we have for this ray from (C′)

$$\Delta(\mu_n\alpha_n\mathbf{d}y_n) = -\frac{m}{2}\left[\frac{\mu_0^2\alpha_0^2}{\mu_n\mu_{n-1}}\kappa_n x^2 + \left(\frac{\mu_n\mu_{n-1}}{\Delta\mu_n}\right)^2 r_n\Delta\left(\frac{\alpha_n}{\mu_n}\right)(\Delta l_n)^2\right];$$

and the curvature of the image formed by secondary foci is given by the equation

$$\frac{1}{R_n'} = \frac{2}{x_n^2}\frac{\partial}{\partial m_n}(\mathbf{d}y_n) = 2\frac{\mu_n^2 a_n}{\mu_0^2 a_0 x^2}\frac{\partial}{\partial m}(\mathbf{d}y_n),$$

and hence the corresponding formulae are

$$\Delta\left(\frac{1}{\mu_n R_n'}\right) = -\{s+t\}\text{ .... (F)},$$

$$\frac{1}{\mu_n R_n'} = -\{S+T\}\text{ .... (F}'\text{)}.$$

The distance between the primary and secondary foci on the chief ray is $x_n^2\left(\frac{1}{2R_n} - \frac{1}{2R_n'}\right)$, *i.e.* $x_n^2 T$ or $\frac{x_n^2}{x^2}(U_2x^2 + 2U_3lx + U_4l^2)$. The radius of the least circle of confusion of the small pencil is proportional to this quantity; and the image will therefore appear distinct to the eye if the value of this expression be small.

241. Ramsden's eye-piece.

As an illustration of these formulae for distortion and curvature, we propose to apply them to the image formed by Ramsden's eye-piece of the micrometer, which is placed at the first principal focus.

The chief ray that enters the eye, placed at the eye-ring, passes before incidence on the eye-piece through the centre of the object-glass of the telescope. As the distance from object-glass to eye-piece is very large compared with the distance of any point of the micrometer from the axis, the error made will be slight if we treat the chief ray as parallel to the axis, and put l zero in formulae (D), (E) and (F). The distortion is therefore measured by $-\frac{1}{2}U_1x^2$, and the distance between the primary and secondary foci on the chief ray by $U_2x_n^2$.

We suppose that the two lenses of the eye-piece are plano-convex, each of numerical focal length f, with their plane faces outwards; and that their distance apart is a, to be determined.

By Cotes' formula the distance z of the micrometer in front of the first lens is $f(f-a)/(2f-a)$.

There are four refracting surfaces separating five media, whose indices of refraction $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4$, are 1, μ, 1, μ, 1 in order. The powers $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ of the four surfaces are 0, $-1/f$, $-1/f$, 0; and the powers K_1, K_2, K_3, K_4 of the systems formed by taking one, two, three or four of the surfaces, are 0, $-1/f$, $-2/f+a/f^2$ and $-2/f+a/f^2$. The auxiliary quantities r_1, r_2, r_3, r_4, which are the heights at which a ray diverging from the first principal focus crosses the surfaces, are given by the equations

$$r_1/a_0 = r_2/a_0 = z = f(f-a)/(2f-a),$$

$$r_3/a_0 = r_4/a_0 = z + a + \kappa_2 za = f^2/(2f-a);$$

and the angles α_1, α_2, α_3, α_4 are given by the equations

$$\mu_1\alpha_1=\mu_0\alpha_0, \quad \mu_2\alpha_2/\mu_0\alpha_0=1+\kappa_2 z=f/(2f-a),$$

$$\alpha_3=\alpha_4=0.$$

If these substitutions be made in (C′) we find, when μ is taken as 3/2, that

$$U_1=-\frac{11a^3-28a^2f+55af^2-32f^3}{9f^4(2f-a)},$$

$$U_2=\frac{7af-8a^2}{f(2f-a)^2}.$$

The distortion vanishes if $a/f=\cdot 864$; the primary and secondary foci coincide and the least circle of confusion reduces to a point if $a/f=\cdot 875$. It seems clear that the distance between the lenses should be more than that ordinarily stated, namely $\frac{2}{3}f$; the actual performance of the eye-piece is however excellent, and it may be that, since the distance of the micrometer from the eye-piece decreases as a/f increases, the distortions of higher order than that considered here will for the larger value of a/f exceed the distortion of the first order for the smaller value.

Achromatism would be more nearly secured by the larger value of a/f than by the smaller.

242. Maxwell's Theorems.

The value of the reduced path between two points near the axis of a symmetrical optical instrument has been applied by Clerk Maxwell to the determination of formulae similar to those of Arts. 239 and 240. They are based on the following theorem:—

Given the reduced distance between two points on the principal focal planes of a symmetrical instrument, to determine its value between any two points.

The principal foci of the instrument being the origins, and the axes of z and z' being drawn away from the instrument, let the reduced path between any point (x_1, y_1) on the first focal plane and any point (x_1', y_1') on the second focal plane be expressed, correct to the fourth order, in the form

$$\begin{aligned} U_1 = U_0 &+ K(x_1x_1'+y_1y_1') \\ &+\tfrac{1}{4}a(x_1^2+y_1^2)^2+\tfrac{1}{2}b(x_1^2+y_1^2)(x_1'^2+y_1'^2)+\tfrac{1}{4}a'(x_1'^2+y_1'^2)^2 \\ &+\{p(x_1^2+y_1^2)+p'(x_1'^2+y_1'^2)\}(x_1x_1'+y_1y_1')+q(x_1x_1'+y_1y_1')^2 \ldots \text{(i)}, \end{aligned}$$

where K is the power of the instrument, and $a, b, \ldots q$ are certain functions of the powers of the refracting surfaces and their distances apart (cf. Art. 188).

The reduced path from a point (x, y, z) to a point (x', y', z') is given by the equation

$$U=\mu\{z^2+(x-x_1)^2+(y-y_1)^2\}^{\frac{1}{2}}+U_1+\mu'\{z'^2+(x'-x_1')^2+(y'-y_1')^2\}^{\frac{1}{2}},$$

$$=\mu z+\mu' z'+\mu\frac{(x-x_1)^2+(y-y_1)^2}{2z}+\mu'\frac{(x'-x_1')^2+(y'-y_1')^2}{2z'}$$

$$-\mu\frac{\{(x-x_1)^2+(y-y_1)^2\}^2}{8z^3}-\mu'\frac{\{(x'-x_1')^2+(y'-y_1')^2\}^2}{8z'^3}+U_1\ldots\text{(ii)}.$$

We have, as in Art. 188, to eliminate x_1, y_1, x_1', y_1' between U and the four equations of type $\dfrac{\partial U}{\partial x_1}=0$.

Let U_4 denote the assemblage of terms of the fourth order in U; then

$$\left.\begin{aligned} Kx_1'+\mu\frac{x_1-x}{z}+\frac{\partial U_4}{\partial x_1}=0 \\ Kx_1+\mu'\frac{x_1'-x'}{z'}+\frac{\partial U_4}{\partial x_1'}=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Solving these equations and substituting the focal lengths f and f' for μ/K and μ'/K, we obtain

$$\left.\begin{aligned} (zz'-ff')\,x_1&=f'(zx'-fx)+\frac{zf}{\mu}\left(f'\frac{\partial U_4}{\partial x_1}-z'\frac{\partial U_4}{\partial x_1'}\right) \\ (zz'-ff')\,x_1'&=f(z'x-f'x')+\frac{z'f'}{\mu'}\left(f\frac{\partial U_4}{\partial x_1'}-z\frac{\partial U_4}{\partial x_1}\right) \end{aligned}\right\}\ldots\text{(iv)},$$

$$\left.\begin{aligned} (zz'-ff')\,(x_1-x)&=z\,(f'x'-z'x)+\frac{zf}{\mu}\left(f'\frac{\partial U_4}{\partial x_1}-z'\frac{\partial U_4}{\partial x_1'}\right) \\ (zz'-ff')\,(x_1'-x')&=z'\,(fx-zx')+\frac{z'f'}{\mu'}\left(f\frac{\partial U_4}{\partial x_1'}-z\frac{\partial U_4}{\partial x_1}\right) \end{aligned}\right\}\ldots\text{(v)}.$$

Now the terms in U of the second order in x and x' are

$$Kx_1x_1'+\mu\frac{(x_1-x)^2}{2z}+\mu'\frac{(x_1'-x')^2}{2z'},$$

i.e.
$$\tfrac{1}{2}\left[x_1\left(Kx_1'+\mu\frac{x_1-x}{z}\right)+x_1'\left(Kx_1+\mu'\frac{x_1'-x'}{z'}\right)\right.$$
$$\left.-\mu x\frac{x_1-x}{z}-\mu' x'\frac{x_1'-x'}{z'}\right]$$

$$=-\tfrac{1}{2}\left[x_1\frac{\partial U_4}{\partial x_1}+x_1'\frac{\partial U_4}{\partial x_1'}+\mu x\frac{x_1-x}{z}+\mu' x'\frac{x_1'-x'}{z'}\right]\ldots\text{(vi)}.$$

If we substitute from (v) in the last two terms of this bracket, they alone give the terms

$$\frac{1}{zz'-ff'}\left\{-(\mu z'x^2 - 2\mu f'xx' + \mu' z x'^2) + fx\left(f'\frac{\partial U_4}{\partial x_1} - z'\frac{\partial U_4}{\partial x_1'}\right) + f'x'\left(f\frac{\partial U_4}{\partial x_1'} - z\frac{\partial U_4}{\partial x_1}\right)\right\}.$$

In this expression the coefficient of $\frac{\partial U_4}{\partial x_1}$ is $f'\frac{(fx - zx')}{zz'-ff'}$ or $-x_1$, since we must neglect the terms of the sixth order in the product of the two factors. It follows that terms of the fourth order are entirely absent from (vi); and hence

$$U = U_0 + \mu z + \mu' z' + \tfrac{1}{2}\frac{\mu z'(x^2+y^2) - 2\mu f'(xx'+yy') + \mu' z(x'^2+y'^2)}{zz'-ff'} + U_4,$$

where in U_4 we have yet to substitute for x_1 and x_1' the first parts only of their values in (iv).

We find that we may write the second approximation to the reduced path between any two points in the form

$$U = U_0 + \mu z + \mu' z' + \tfrac{1}{2}\frac{\mu z'(x^2+y^2) - 2\mu f'(xx'+yy') + \mu' z(x'^2+y'^2)}{zz'-ff'}$$
$$+ \tfrac{1}{4}A(x^2+y^2)^2 + \tfrac{1}{2}B(x^2+y^2)(x'^2+y'^2) + \tfrac{1}{4}A'(x'^2+y'^2)^2$$
$$+ \{P(x^2+y^2) + P'(x'^2+y'^2)\}(xx'+yy') + Q(xx'+yy')^2 \ldots \text{(vii)}.$$

If we put $z = \lambda f$, $z' = \lambda' f'$, $D = \lambda\lambda' - 1$, the coefficients are given by the equations

$$A \,.\, D^4 = -\frac{1}{2}\frac{\mu}{f^3}\lambda\lambda'^4 - \frac{1}{2}\frac{\mu'}{f'^3}\lambda' + a + 2b\lambda'^2 + a'\lambda'^4 - 4p\lambda' - 4p'\lambda'^3 + 4q\lambda'^2,$$

$$A' \,.\, D^4 = -\frac{1}{2}\frac{\mu}{f^3}\lambda - \frac{1}{2}\frac{\mu'}{f'^3}\lambda^4\lambda' + a\lambda^4 + 2b\lambda^2 + a' - 4p\lambda^3 - 4p'\lambda + 4q\lambda^2,$$

$$B \,.\, D^4 = -\frac{1}{2}\frac{\mu}{f^3}\lambda\lambda'^2 - \frac{1}{2}\frac{\mu'}{f'^3}\lambda^2\lambda' + a\lambda^2 + b(1+\lambda^2\lambda'^2) + a'\lambda'^2 - 2(p\lambda + p'\lambda')(1+\lambda\lambda') + 4q\lambda\lambda',$$

$$P \,.\, D^4 = \frac{1}{2}\frac{\mu}{f^3}\lambda\lambda'^3 + \frac{1}{2}\frac{\mu'}{f'^3}\lambda\lambda' - a\lambda - b\lambda'(1+\lambda\lambda') - a'\lambda'^3 + p(1+3\lambda\lambda') + p'\lambda'^2(3+\lambda\lambda') - 2q\lambda'(1+\lambda\lambda'),$$

$$Q \,.\, D^4 = -\frac{1}{2}\frac{\mu}{f^3}\lambda\lambda'^2 - \frac{1}{2}\frac{\mu'}{f'^3}\lambda^2\lambda' + a\lambda^2 + 2b\lambda\lambda' + a'\lambda'^2 - 2(p\lambda + p'\lambda')(1+\lambda\lambda') + q(1+\lambda\lambda')^2.$$

These coefficients appear to be infinite if $\lambda\lambda' = 1$, *i.e.* if the points (x, y, z), (x', y', z') lie on conjugate planes, but the equations from which they are deduced are then really indeterminate, and the reduced path takes another form as in Art. 190.

We may notice in connection with the term S in the expression for the curvature of the image given below in Art. 246 that

$$(B - Q)\,D^2 = (b - q).$$

243. Orthotomic surfaces.

Let a point $(x, 0, \lambda f)$ be the origin of a pencil of light; that orthotomic surface of the emergent pencil which passes through the second principal focus is given by the equation

$$U_0 + \mu\lambda f + \mu' z' + \tfrac{1}{2}\frac{\mu'\lambda f(x'^2 + y'^2) - 2\mu' f x x' + \mu z' x^2}{\lambda f z' - f f'}$$
$$+ \tfrac{1}{4} a x^4 + \tfrac{1}{2} B_0 x^2 (x'^2 + y'^2) + \tfrac{1}{4} A_0' (x'^2 + y'^2)^2$$
$$+ \{P_0 x^2 + P_0' (x'^2 + y'^2)\}\, x x' + Q_0 x^2 x'^2 = U_0 + \mu\lambda f + \tfrac{1}{4} a x^4,$$

where $A_0' \ldots Q_0$ are the values of $A' \ldots Q$, when λ' is zero.

Substituting the first approximation for z' in the denominator $(\lambda z' - f')$, the equation of this orthotomic surface may be written

$$z' = \tfrac{1}{2}\left\{\frac{\lambda}{f'} + x^2\left(\frac{5}{2}\frac{\lambda}{f'^3} - \frac{B_0}{\mu'} - \frac{2Q_0}{\mu'}\right)\right\} x'^2$$
$$+ \tfrac{1}{2}\left\{\frac{\lambda}{f'} + x^2\left(\frac{1}{2}\frac{\lambda}{f'^3} - \frac{B_0}{\mu'}\right)\right\} y'^2 - \left\{\frac{1}{f'} + x^2\left(\frac{1}{2f'^3} + \frac{P_0}{\mu'}\right)\right\} x x'$$
$$+ \tfrac{1}{4}\left(\frac{\lambda^3}{f'^3} - \frac{A_0'}{\mu'}\right)(x'^2 + y'^2)^2 - \left(\frac{\lambda^2}{f'^3} + \frac{P_0'}{\mu'}\right) x x' (x'^2 + y'^2).$$

To a first approximation this surface is necessarily a sphere with its centre at the point $(x/\lambda, 0, f'/\lambda)$; but by differentiation we may find the two principal radii of curvature at any point of this surface correct to squares of the small quantities x, x' and y', *i.e.* we may find the positions of the primary and secondary foci on the ray through any point of this surface.

The surface so determined through F_2 becomes indeterminate when the incident pencil consists of parallel rays. In that case we must draw the orthotomic surface through any other point on the axis, for instance, through the second unit point for which $\lambda' = 1$.

244. The Aberration formula.

Let a pencil of rays diverge from a point on the axis at distance λf in front of the first principal focus F_1; that orthotomic surface of the emergent rays which passes through F_2 is given by the equation

$$U_0 + \mu\lambda f + \mu' z' + \tfrac{1}{2}\mu' \frac{x'^2 + y'^2}{z' - f'/\lambda} + \tfrac{1}{4} A_0' (x'^2 + y'^2)^2 = U_0 + \mu\lambda f,$$

where A_0' is the value of A' when λ' is zero,

i.e. $$A_0' = -\frac{1}{2}\frac{\mu}{f^3}\lambda + a\lambda^4 + 2(b + 2q)\lambda^2 + a' - 4p\lambda^3 - 4p'\lambda.$$

This surface is one of revolution, and its equation may be written as

$$z' = \tfrac{1}{2}\frac{\lambda}{f'}(x'^2 + y'^2) + \tfrac{1}{4}\left(\frac{\lambda^3}{f'^3} - \frac{A_0'}{\mu'}\right)(x'^2 + y'^2)^2.$$

The normal to this surface at a point distant x' from the axis of the instrument meets it at a point at distance

$$\frac{f'}{\lambda} + \left(\frac{A_0'}{\mu'} - \frac{1}{2}\frac{\lambda^3}{f'^3}\right)\frac{f'^2 x'^2}{\lambda^2}$$

behind F_2, *i.e.* the longitudinal aberration for this ray is the second term in this expression.

Also if α_0, α' be the initial and final angles of divergence of this ray from the axis, we have approximately $\alpha' = \lambda x'/f'$, and $\mu_0\alpha_0 = \mu'\alpha'/\lambda = \mu_0\alpha' f'/\lambda f$. Hence we may write the aberration formula as

$$\mu'\alpha'^2 \mathbf{d}v = f^4\alpha_0^4\left(A_0' - \frac{\mu'}{2}\frac{\lambda^3}{f'^3}\right).$$

By comparison with the formula (i) of Art. 143, when expressed in terms of α_0 and of λf, the distance of the origin of light from F_1 we may determine in any system whatever the values of a, a', p, p' and $(b + 2q)$. Also, as we shall see below, we have in any case the equation

$$2(b - q) = K^2 \sum_1^n \kappa_r/\mu_r\mu_{r-1};$$

so that b and q may be determined separately.

In this way, on making the substitutions $\mu'\alpha'/\mu\alpha = \lambda, y/\alpha = (\lambda - 1)f$ in the original formula (A) of Art. 141, we find that for a single

spherical surface of power κ separating media of indices μ and μ'

$$a = (\mu/\mu')^2, \quad a' = (\mu'/\mu)^2, \quad p = (\mu/\mu'), \quad p' = (\mu'/\mu),$$
$$b = (\mu^2 - \mu\mu' + \mu'^2)/\mu\mu', \quad q = 1,$$

all multiplied by a common factor $\dfrac{1}{2}\dfrac{\kappa^3}{(\mu' - \mu)^2}$.

In the case of a thin lens of power K and index μ, the powers of the surfaces being κ_1 and κ_2, we substitute

$$\alpha_2 = \lambda\alpha_0, \quad \mu\alpha_1 = (\lambda\kappa_1 + \kappa_2) f\alpha_0, \quad y = (\lambda - 1) f\alpha_0$$

in (I) Art. 146, and find for the coefficients in the reduced path

$$a = (\mu^3 - 2\mu^2 + 2)\,\kappa_1^2 + (2\mu^3 - 2\mu^2 - \mu)\,\kappa_1\kappa_2 + \mu^3\kappa_2^2,$$
$$p = \kappa_1^2 + (\mu^2 - \mu - 1)\,\kappa_1\kappa_2 + \mu^2\kappa_2^2,$$
$$b = (\mu^2 - \mu + 1)(\kappa_1^2 + \kappa_2^2) + (2\mu^2 - 3\mu)\,\kappa_1\kappa_2,$$
$$q = \mu(\kappa_1^2 + \kappa_2^2) + (\mu - 2)\,\kappa_1\kappa_2,$$

all multiplied by a common factor $\dfrac{1}{2}\dfrac{K}{\mu(\mu - 1)^2}$.

245. The Distortion formula.

Let (l, m, n) be the direction-cosines of an incident ray, which meets the first focal plane in (x_1, y_1), and let (l', m', n') be the direction-cosines of the emergent ray, meeting the second focal plane in (x_1', y_1'). These direction-cosines are given by the equations $\mu' l' = \dfrac{\partial U_1}{\partial x_1'}$, $\mu l = -\dfrac{\partial U_1}{\partial x_1}$, where U_1 has the value given in (i) Art. 242.

Hence

$$\mu' l' = Kx_1 + a'(x_1'^2 + y_1'^2)\,x_1' + b\,(x_1^2 + y_1^2)\,x_1'$$
$$+ \{p\,(x_1^2 + y_1^2) + p'(x_1'^2 + y_1'^2)\}\,x_1 + 2\,(p'x_1' + qx_1)(x_1x_1' + y_1y_1') \qquad \text{............(i)},$$

$$-\mu l = Kx_1' + a\,(x_1^2 + y_1^2)\,x_1 + b\,(x_1'^2 + y_1'^2)\,x_1$$
$$+ \{p\,(x_1^2 + y_1^2) + p'(x_1'^2 + y_1'^2)\}\,x_1' + 2\,(px_1 + qx_1')(x_1x_1' + y_1y_1') \qquad \text{............(ii)}.$$

In the terms of the third degree in (i) we may substitute the first approximations $x_1' = -lf$, $y_1' = -mf$. We then obtain the direction-cosines of the emergent ray, correct to the third order terms, which the first method of this chapter failed to give.

If in (ii) we make the same substitutions, we obtain x_1' in terms of x_1, y_1, l and m to the third order. Now if the incident ray originate in a point of coordinates (x, y, z), the coordinates (x_1, y_1) are given by the equations

$$x_1 - x = lr, \quad y_1 - y = mr, \quad z = \{1 - \tfrac{1}{2}(l^2 + m^2)\}\, r,$$

or $x_1 = x + lz\{1 + \frac{1}{2}(l^2 + m^2)\}, \quad y_1 = y + mz\{1 + \frac{1}{2}(l^2 + m^2)\}$...(iii).

Similarly if the emergent ray pass through the point (x', y', z')

$$x' = x_1' + l'z'\{1 + \tfrac{1}{2}(l'^2 + m'^2)\}, \quad y' = y_1' + m'z'\{1 + \tfrac{1}{2}(l'^2 + m'^2)\} \quad \text{............(iv)}.$$

If we substitute in (iv) the values of x_1', y_1', l', m', expressed in terms of x_1, y_1, l, m, and further substitute for (x_1, y_1) from (iii), we have the coordinates of the point in which the emergent ray cuts any plane. If z' be taken equal to ff'/z, this will be the plane of the geometrical foci, and x' will differ from xf/z by terms of the third order only in x, y, l, m. We shall have obtained the integral of the difference-equation (C′) of Art. 238; but as the quantities $U_1, \ldots U_4$ of that integral are now expanded in powers of z, the coefficients appear more complicated.

(In Maxwell's method the coordinate z is the distance of the origin of light in front of the first principal focus; in the method given in the first part of this chapter we used z for its distance in front of the first refracting surface.)

246. Curvature of Image.

As before, we confine our attention to the primary and secondary foci of a small pencil whose chief ray passes in an axial plane, which we take as the plane of reference xz. Let (l, m, n) be the direction-cosines of any ray incident on the first focal plane at (x_1, y_1), and let (l', m', n') be those of the emergent ray meeting the second focal plane at (x_1', y_1'). We have the equations

$$\mu' l' = \frac{\partial U_1}{\partial x_1'}, \quad \mu' m' = \frac{\partial U_1}{\partial y_1'}, \quad -\mu l = \frac{\partial U_1}{\partial x_1}, \quad -\mu m = \frac{\partial U_1}{\partial y_1} \quad \text{......(i)}.$$

Let a ray pass in the plane zx and meet a consecutive ray in this plane in the point $(x, 0, z)$, and let v_1 be the distance along the ray from this point to the first focal plane; then

$$l = \frac{x_1 - x}{v_1}, \quad \frac{\partial l}{\partial x_1} = \frac{1}{v_1} - \frac{(x_1 - x)^2}{v_1^3} = \frac{1 - l^2}{v_1} \quad \text{...........(ii)}.$$

Let these two rays on emergence intersect in the primary focus at distance v_1' along the ray from the second focal plane; then

$$l' = \frac{x' - x_1'}{v_1'}, \quad \frac{\partial l'}{\partial x_1'} = -\frac{1 - l'^2}{v_1'} \quad \text{..................(iii).}$$

Hence, differentiating (i), we have

$$\left.\begin{aligned} \mu \frac{1 - l^2}{v_1} + \frac{\partial^2 U_1}{\partial x_1^2} + \frac{\partial^2 U_1}{\partial x_1 \partial x_1'} \frac{\partial x_1'}{\partial x_1} = 0 \\ \mu' \frac{1 - l'^2}{v_1'} + \frac{\partial^2 U_1}{\partial x_1 \partial x_1'} \frac{\partial x_1}{\partial x_1'} + \frac{\partial^2 U_1}{\partial x_1'^2} = 0 \end{aligned}\right\} \text{..............(iv),}$$

and therefore

$$\left(\mu \frac{1 - l^2}{v_1} + \frac{\partial^2 U_1}{\partial x_1^2}\right)\left(\mu' \frac{1 - l'^2}{v_1'} + \frac{\partial^2 U_1}{\partial x_1'^2}\right) = \left(\frac{\partial^2 U_1}{\partial x_1 \partial x_1'}\right)^2 \text{......(v).}$$

In the same way, considering two rays, one of which is the chief ray and lies throughout in the axial plane, and the other makes a small angle with it in the perpendicular plane, we have for the latter ray

$$m = \frac{y_1}{v_2}, \quad \frac{\partial m}{\partial y_1} = \frac{1}{v_2},$$

where y_1 is made zero after differentiation; and on emergence

$$m' = \frac{y_1'}{v_2'}, \quad \frac{\partial m'}{\partial y_1'} = \frac{1}{v_2'}.$$

Hence from equations similar to (iv) we deduce

$$\left(\frac{\mu}{v_2} + \frac{\partial^2 U_1}{\partial y_1^2}\right)\left(\frac{\mu'}{v_2'} + \frac{\partial^2 U_1}{\partial y_1'^2}\right) = \left(\frac{\partial^2 U_1}{\partial y_1 \partial y_1'}\right)^2 \text{............(vi),}$$

where y_1 and y_1' are made zero after differentiation.

Now let $z\,(=\lambda f)$ be the distance of a point on the axis in front of F_1, $z'\,(=f'/\lambda)$ the distance of its geometrical focus behind F_2, and let R_1, R_1' be the radii of curvature of a linear object in the plane zx and its image formed by primary foci, both reckoned positive if convex to the eye. Then

$$v_1 n = \lambda f + x^2/2R_1, \quad v_1 l = x_1 - x,$$
$$v_1' n' = f'/\lambda - x'^2/2R_1', \quad v_1' l' = x' - x_1'.$$

Hence
$$\frac{1 - l^2}{v_1} = \frac{1}{\lambda f}\left\{1 - \frac{x^2}{2R_1 \lambda f} - \frac{3}{2} l^2\right\},$$
$$\frac{1 - l'^2}{v_1'} = \frac{\lambda}{f'}\left\{1 + \frac{\lambda x'^2}{2R_1' f'} - \frac{3}{2} l'^2\right\}.$$

Also $$\frac{\partial^2 U_1}{\partial x_1^2} = 3(ax_1^2 + 2px_1x_1' + qx_1'^2) + (b-q)x_1'^2 = 3\alpha + (b-q)x_1'^2,$$

$$\frac{\partial^2 U_1}{\partial x_1'^2} = 3(qx_1^2 + 2p'x_1x_1' + a'x_1'^2) + (b-q)x_1^2 = 3\beta + (b-q)x_1^2,$$

$$\frac{\partial^2 U_1}{\partial x_1 \partial x_1'} = K + 3\{px_1^2 + (b+q)x_1x_1' + p'x_1'^2\} - (b-q)x_1x_1'$$
$$= K + 3\gamma - (b-q)x_1x_1'.$$

If we substitute these values in (v) and make use also of the relation for linear magnification, $x' = x/\lambda$, we obtain

$$\frac{1}{\mu' R_1'} - \frac{1}{\mu R_1} = -\frac{2(b-q)}{K^2}\frac{(x_1+\lambda x_1')^2}{x^2}$$
$$-\frac{3}{K^2x^2}\{2(\alpha\lambda^2 - 2\gamma\lambda + \beta) - (l^2 + l'^2)\lambda K\} \text{......(vii).}$$

Similarly, to obtain the curvature of the image formed by secondary foci, we must substitute in (vi) from the equations

$$\frac{1}{v_2} = \frac{1}{\lambda f}\left\{1 - \frac{x^2}{2R_2\lambda f} - \frac{1}{2}l^2\right\},\quad \frac{1}{v_2'} = \frac{\lambda}{f'}\left\{1 + \frac{\lambda x'^2}{2R_2'f'} - \frac{1}{2}l'^2\right\},$$

$$\left.\begin{aligned}
\frac{\partial^2 U_1}{\partial y_1^2} &= ax_1^2 + 2px_1x_1' + bx_1'^2 = \alpha + (b-q)x_1'^2 \\
\frac{\partial^2 U_1}{\partial y_1'^2} &= bx_1^2 + 2p'x_1x_1' + a'x_1'^2 = \beta + (b-q)x_1^2 \\
\frac{\partial^2 U_1}{\partial y_1 \partial y_1'} &= K + px_1^2 + 2qx_1x_1' + p'x_1'^2 = K + \gamma - (b-q)x_1x_1'
\end{aligned}\right\}.$$

Hence

$$\frac{1}{\mu' R_2'} - \frac{1}{\mu R_2} = -\frac{2(b-q)}{K^2}\frac{(x_1+\lambda x_1')^2}{x^2}$$
$$-\frac{1}{K^2x^2}\{2(\alpha\lambda^2 - 2\gamma\lambda + \beta) - (l^2 + l'^2)\lambda K\} \text{......(viii).}$$

Moreover, since to the first approximations $x_1 = l'f'$, $x_1' = -lf$ and $x_1 - x = l\lambda f$, we have $x_1 + \lambda x_1' = x$; and therefore (vii) and (viii) may be written as

$$\frac{1}{\mu' R_1'} - \frac{1}{\mu R_1} = -(S + 3T) \text{ .... (E')},$$

$$\frac{1}{\mu' R_2'} - \frac{1}{\mu R_2} = -(S + T) \text{ .... (F')},$$

where $$S = 2(b-q)/K^2,$$

and $$T = \frac{1}{K^2x^2}\{2(a\lambda^2 - 2\gamma\lambda + \beta) - (l^2 + l'^2)\lambda K\}$$

$$= \frac{1}{K^2x^2}\{2(a\lambda^2 - 2p\lambda + q)x_1^2 + 2(a' - 2p'\lambda + q\lambda^2)x_1'^2$$

$$+ 4(p\lambda^2 - \overline{b+q}\lambda + p')x_1x_1' - (l^2 + l'^2)\lambda K\}.$$

We see then that in any case the curvature of the image depends on the quantities S and T. The first of these is independent of the position of the object, and by comparison with (E′) Art. 240 we deduce

$$2(b-q) = K^2\sum_1^n \frac{\kappa_r}{\mu_r\mu_{r-1}};$$

the second is here expressed in terms of x and of x_1 and x_1', which are connected with the origin of light and the direction of the incident ray by the equations $x_1' = -lf$, and $x_1 = x + l\lambda f$. We may either regard these formulae as giving the curvatures of the final images after refraction through an entire coaxial system, or we may use them as difference-equations, and substitute the successive values of the coordinates x and λf of the geometrical focus and the successive values of the direction-cosine l of the chief ray, the values of the coefficients $a \ldots q$ being those given in Art. 244 either for a single refraction, or for refraction through a thin lens as the case may be.

EXAMPLES.

1. A small object is placed perpendicularly to the axis of a single refracting spherical surface at the aplanatic point; the curvatures of the images formed by primary and secondary foci are both κ/μ, the function T vanishing for this point.

2. A pencil of rays diverging from a point of coordinates $(x, 0, \lambda f)$ passes through a coaxial system; the principal radii of curvature at the second principal focus of the orthotomic surface through that point are

$$\frac{f'}{\lambda}\left\{1 + x^2\left(\frac{B_0 + 2Q_0}{\mu}\frac{f}{\lambda} - \frac{1}{f'^2}\right)\right\},$$

and $$\frac{f'}{\lambda}\left\{1 + x^2\frac{B_0}{\mu}\frac{f}{\lambda}\right\}.$$

3. A pencil of parallel rays making a small angle l with the axis is incident on a refracting coaxial system; the principal radii of curvature at the second unit point of the orthotomic surface of the emergent light through that point are

$$f'\left\{1+l^2\left(\frac{1}{2}\frac{\mu^2}{\mu'^2}-\frac{3a-6p+b+2q}{\mu}f^3\right)\right\},$$

and

$$f'\left\{1+l^2\left(\frac{1}{2}\frac{\mu^2}{\mu'^2}-\frac{a-2p+b}{\mu}f^3\right)\right\},$$

where $a \ldots q$ are the coefficients of the terms of the fourth order in the reduced path.

4. A pencil of parallel rays inclined to the axis is incident on a reflecting telescope. Shew that with the same aperture the deviation of the extreme emergent rays from the direction of the axis of the emergent pencil is less in Cassegrain's telescope than in Gregory's.

INDEX.

The numbers refer to the pages.

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